# G13GAM — Game Theory

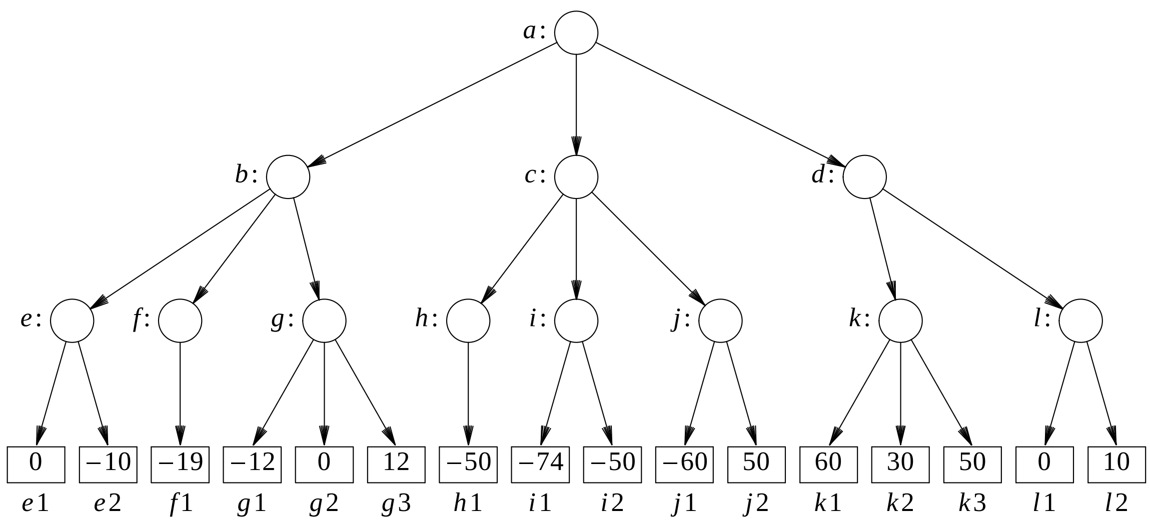
The following questions are mostly taken from old exam papers, except that virtually all bookwork has been eliminated. A very large majority of all other questions ever set for this module are included, though sometimes updated; where quite similar questions were set, they have usually either been somehow combined or else one version discarded. Questions marked † constitute significantly less than a complete exam question (no question is [intended to be!] significantly *more*!). Those marked ‡ were intended, under the pre-modularisation examination system, also to be somewhat routine in nature.

Questions marked **2001** (for example) are taken (sometimes slightly modified) from the exam of that year. The 2007 exam paper is omitted, in case you want to use it as a 'mock' exam; the 2008 paper is not yet available! At the time of writing, the 2007 paper is not yet available on the University's *past paper* site; the *draft* paper [better than the real thing!] together with solutions and feedback is available in a link near the bottom of the module web page.

Details of what questions should be handed in and when will be given as the module progresses. Solutions to (virtually) all of these questions will be given out as appropriate, amounting almost to a textbook on the module.

ANW

**1** **1991** Suppose that, in the following game tree, static evaluation would (if invoked) yield the values given in the square boxes at the leaf nodes. Assume that, perhaps following iterative deepening, the tree is to be searched in the obvious top–bottom, left–right ordering.




(*a*) Evaluate all the internal nodes, showing that the root node has value 50. [CHECK: Nodes *b* and *e* have values −10 and 10, respectively.]

(*b*) Suppose that $\alpha$-$\beta$ pruning is used with initial bounds $-\infty$ and $+\infty$. Indicate clearly which nodes are visited and which are pruned in order to establish the value of the root node.

(*c*) Suppose instead that the initial $\alpha$-$\beta$ bounds are both 40. Which nodes are now visited and which pruned? Indicate clearly what deductions can be made at each internal node visited.

**2** **1994+2005** A position evaluator has been invoked at a node with alpha-beta levels $\alpha$ and $\beta$. It internally sets its levels at $\alpha'$ and $\beta'$, where $\alpha < \alpha' < \beta' < \beta$, and invokes itself recursively to evaluate its first sub-node; the evaluation returns the value $\gamma$. [All levels and values measured from the point of view of the player to move at *this* node; assume that levels and values have been correctly negated when being passed to and returned from the sub-node.] For each of the cases

$$(a)\ \gamma < \alpha;\ (b)\ \alpha < \gamma < \alpha';\ (c)\ \alpha' < \gamma < \beta';\ (d)\ \beta' < \gamma < \beta;\ \text{and } (e)\ \beta < \gamma,$$

explain what can be deduced about the value of the current node, and in particular what changes may safely be made to the levels, whether pruning is possible, and whether a re-search may be necessary.

**3** **2000†** Construct a game tree with the following specification. Each node has 3 children. A depth-limited dynamic evaluation is to be made of the root, limited to depth 3, so that 27 static evaluations are made in the absence of pruning. Each static evaluation returns an integer value; the values are to be such that the value of the root is 0.

For your constructed tree, indicate which nodes are pruned if $\alpha$-$\beta$ pruning is used with initially $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$.

**4** **1995+2003** A game-playing computer can perform static analysis on approximately 1 000 000 nodes per second. In a given position, suppose that White is to move, and has a winning position; suppose further that in winning positions there are typically 40 possible moves for the side to move, and in losing positions only 20. Assume that the time taken in activities other than static analysis can be ignored. Estimate the time taken to perform a full search to 10 ply (5 moves by each side) in the cases: (*a*) with no $\alpha$-$\beta$ pruning; (*b*) with $\alpha$-$\beta$ pruning but no move-ordering; and (*c*) with $\alpha$-$\beta$ pruning and the best possible move-ordering. Comment briefly on how good move-ordering can be achieved.

**5** **1997** In Othello, we may assume, as an approximation, that every game lasts for exactly 60 moves, and that when there are $n$ moves to go [*i.e.* after $60-n$ moves] there are $n$ empty squares on the board, any one of which may be chosen for a move. When the board is full, after move 60, a simple counting process determines the winner; assume that a computer program can complete this process in about 100 microseconds, and that by comparison the time taken to play or unplay moves can be ignored. This program is determining the 49th move [so there are 12 empty squares]. *Estimate* how long the program will take to complete its analysis in the cases:
(*a*) when no pruning is carried out;
(*b*) when the program uses $\alpha-\beta$ pruning, is able to use perfect move ordering, and is in a winning position;
(*c*) as (*b*), but in a losing position; and
(*d*) as (*b*), but with no move-ordering information available, and assuming that in a winning position it is roughly an evens chance whether any given move wins or loses.

**6** **2001** The length of a game $G$ is defined to be 0 if $G$ has no options and otherwise to be $1+l$ where $l$ is the length of the longest option (for either Left or Right) of $G$.

*(a)* Let $m_n$ be the number of games of length $\leqslant n$. Show that $m_n$ satisfies $m_{n+1} = 2^{2m_n}$, and deduce that there are 252 games of length (exactly) 2. By considering which lists of options contain dominated options, or otherwise, show that these 252 games can have no more than 36 distinct values, and show further that 15 of them have the value 0.

*(b)* Show that there are $2^n$ distinct new numbers of length $n$, and write down a number $x$ such that $1 < x < 2$ and $x$ has length 10.

**7** Consider the Noughts-and-Crosses position

| | | |
|---|---|---|
| O | | X |
| X | O | |
| O | | |

(with Cross to move). Evaluate this position using $\alpha-\beta$ pruning to prune as many nodes as possible, drawing a tree showing which nodes have been visited and what their values are. (For finding the strongest move, assume that completing a line-of-three is best, followed by blocking an opposing line-of-three, otherwise play at random.) How many nodes would be visited if a full search of the tree were made?

**8** [based on:] **1988+1994+1999** For any game $G$, the game $+_G$ ('tiny-G') is defined to be $\{0 \mid \{0 \mid -G\}\}$. Show that $0 < +_G \leqslant \frac{1}{2}$ for every $G$, and write down a $G$ for which $+_G = \frac{1}{2}$. Show that if $G$ is a negative number, then $+_G$ is a positive *number*, and find this number in the cases $G = -1$ and $G = -\pi$. Show that if $G$ is a positive number, then $+_G < \uparrow$, where $\uparrow = \{0|*\}$.

**9** **1993+2005** Left and Right are each driving sheep along a mountain path. Left has two sheep and Right has three. They meet at a point where it is not possible to squeeze past or turn round. Each Left sheep is moving to the right, and can either move one space or can jump over one Right sheep into an empty space immediately beyond; similarly, each Right sheep can move one space to the left or can jump over one Left sheep. When they meet, Left's sheep are separated by one space, as suggested by the figure, where the herdsmen (who do not move) are standing at the positions marked 'X'. Show that this position has value tiny-¼, where, as usual, tiny-$x$ is $\{0|\{0|-x\}\}$, and the herdsman who first cannot move a sheep has to give way.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| X | L | | L | R | R | R | X |

**10** **2004** In the game Sylver Coinage, Left and Right take turns to nominate a new coin, worth an integer number of pennies. Each new coin must be independent of its predecessors, so that it cannot be made up from multiples of them. The player who is forced to nominate the 1p coin loses.

*(a)* Show that the game may last for an arbitrarily large number of moves, but, by considering common factors of coins already nominated or otherwise, that it must terminate. [You may quote Sylvester's theorem, that if the positive integers $a$ and $b$ have no common factors, then every integer greater than or equal to $(a-1)(b-1)$ can be represented as $pa+qb$ where $p$ and $q$ are non-negative integers.]

*(b)* Show that the first player to nominate either the 2p coin or the 3p coin loses. Left starts by nominating the 4p coin and Right responds by nominating the 5p coin. Determine who should win this position.

**11** **1989‡** Explain carefully what is meant by $A+B$ and by $A=B$, where $A$ and $B$ are games. Show that $1+1=2$, where $1=\{0|\}$ and $2=\{1|\}$. [Do not assume without proof that $x+0=x$.]

**12** **1997+1989** An *all-small* game is one in which if either player can move then so can the other, and in which every possible move is also to an all-small game. If $G$ is an all-small game, show that $G<1$. Deduce that in fact $G<x$ for every positive number $x$.

Give an example of a strictly positive all-small game. Also give an example of a strictly positive non-all-small game which is smaller than every positive number.

**13** **†** Let $x$ and $y$ be numbers such that $x>y\geqslant 0$. Show that $+_x < +_y$ [where, as usual, the game $+_z$ ('tiny-z') is defined to be $\{0 \mid \{0 \mid -z\}\}$] and deduce that $n\cdot +_x < +_y$ [where $n\cdot G$ means $G+G+\ldots+G$, $n$ terms] for any integer $n$.

**14** **1991** If $A$ and $B$ are games, explain briefly what are meant by $-A$, $A+B$, $A>B$ and $A \parallel B$.

For each of the following relations, either find games $A>0$ and $B\parallel 0$ such that the relation holds, or prove that no such games exist:

$$(a)\ \ A+B>0;\quad (b)\ \ A+B=0;\quad (c)\ \ A+B\parallel 0;\quad (d)\ \ A+2B=0;\quad (e)\ \ A+2B\parallel 0.$$

If a game $G$ has a left option $L$, then $L-G$ is said to be a *left incentive* of $G$. Similarly, if $G$ has a right option $R$, then $G-R$ is a *right incentive*. Show that if all incentives [both left and right] of $G$ are less than $-\varepsilon$, where $\varepsilon$ is a strictly positive *number*, then $G$ is a number. If $\varepsilon=2^{-n}$, where $n$ is a positive integer, deduce that $G=k\varepsilon$ for some integer $k$.

**15** **2000** Explain what is meant by $G+H$ when $G$ and $H$ are games. Show that $(a)$ $G+0=G$ for every game $G$; $(b)$ $G+(-G)=0$ for every game $G$; $(c)$ $G+G=0$ for every impartial game $G$; $(d)$ $\frac{1}{2}+\frac{1}{2}=1$, treating $\frac{1}{2}$ and 1 as games; $(d)$ $\uparrow+\uparrow=\{0|\uparrow\}+*$, where, as usual, $\uparrow=\{0|*\}$.

**16** **1996** Suppose that $p$ and $q$ are numbers. Explain how to evaluate the game $\{p|q\}$. Write down the values of the games $\{1|3\}$, $\{1|4\}$, $\{-1|4\}$, $\{-1|-1\}$, $\{-1|-4\}$, $\{-1|0\}$, $\{\frac{1}{3}|\pi\}$ and $\{\frac{7}{16}|\frac{15}{16}\}$.

What happens if $p$ and $q$ are not necessarily numbers, but nevertheless $p<q$? Illustrate your account by considering the cases $\{\uparrow|1\}$ [where $\uparrow$ is the positive but infinitesimal game $\{0|*\}$], $\{0|\uparrow\}$ [ $=\uparrow+\uparrow+*$] and $\{\pm 1|2\}$.

**17** **1998**

$(a)$ Let $G$ be the game $\{A,B,C,\ldots \mid D,E,F,\ldots\}$, and let $H$ be the game which is the same as $G$ but with Left's option $A$ omitted.
$(i)$ Show that $G\geqslant H$.
$(ii)$ Show that if $A\leqslant B$ then $G=H$.
$(iii)$ Suppose that $A$ is strictly greater than all Left options of $H$. Does it follow that $G>H$? Give reasons.

$(b)$ Let $p$ be a number. Show that $p+*=\{p \mid p\}$. Does this result still hold if $p$ is replaced by a general game, $G$? Give reasons.

**18** **1989** Explain the terms *hot* and *cold* as applied to positions in games. Give an example of a hot game (in which positions are typically hot) and an example of a cold game. What strategy should you follow when playing the disjunctive sum of various games of differing temperatures?

Let $G$ be the game $\{2|7\}+\{7|2\}+\{1|-9\}+\{0|-2\}$. Show that, with best play, $G$ eventually becomes the number 6 with Left to play, or $-1$ with Right to play.

Show that, nevertheless, $G \neq H$, where $H$ is $\{6|-1\}$. What is the outcome of the game $G-H$? Write down a game $K$ such that $G+K$ and $H+K$ have different outcomes. Evaluate $2G$, and verify that $2G = 2H$.

**19** **2004†** A *number* game is a finite two-player game in normal form in which all options are numbers. Show that if $H$ is a disjunctive sum of number games, then $2H$ is a number. Find this number in the case where

$$H = \{1,2,3 \mid 6,7,8\}+\{1,2,3 \mid 1,2,3\}+\{8,7,6 \mid 3,2,1\}+\{-4 \mid 4\}+\{-4 \mid -4\}+\{4 \mid -4\}.$$

**20** **2002** If $G$ is a game of form $G = \{A,B,C,\ldots \mid D,E,F,\ldots\}$, and similarly $H$ has form $H = \{P,Q,R,\ldots \mid S,T,U,\ldots\}$, then the *ordinal sum* $G{:}H$ of $G$ and $H$ is defined by

$$G{:}H = \{A,B,C,\ldots,\ G{:}P, G{:}Q, G\,R,\ldots \mid D,E,F,\ldots,\ G{:}S, G{:}T, G\,U,\ldots\};$$

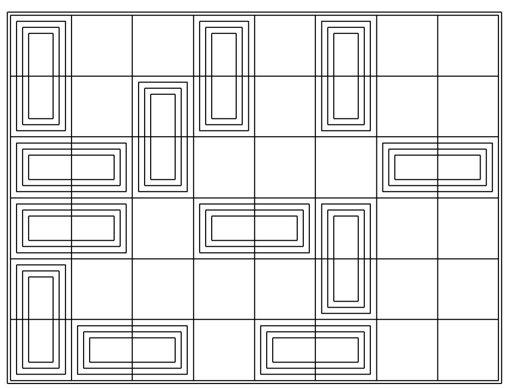
that is, each move in $G$ destroys $H$ while moves in $H$ leave $G$ unchanged. Show that $1{:}{-1} = \frac{1}{2}$, that $*{:}{*n} = *(n+1)$ [where $*n$ represents, as usual, a Nim-heap of size $n$ and $* = *1 = \{0|0\}$], and find $*{:}1$.

Show, by considering strategies in $G{:}H - G{:}K$ or otherwise, that if $H \geqslant K$ then $G{:}H \geqslant G{:}K$. Deduce that if $H = K$ then $G{:}H = G{:}K$. By considering the case $G = *$, deduce the Tree Principle of impartial Hackenbush.

**21** **1995†+2003†** Reduce to simplest terms the games

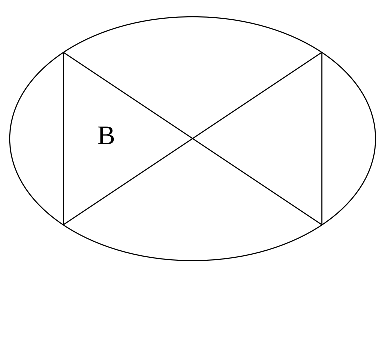
($a$) $\uparrow + *$; ($b$) $\{\uparrow \mid \uparrow\} + \downarrow$; ($c$) $\{0, *, *3, *5 \mid 0, *, *4, *6\}$; ($d$) $2 + -_2$; and ($e$) $\{\downarrow \mid \uparrow\} + \{4 \mid 0\} + \pm 2 + 2*$.

**22** **1998** Evaluate the position shown from a position in Domineering. In particular, determine who should win with best play *(a)* if it is Left, playing the fLat dominoes, to play; and *(b)* if it is Right, playing the upRight dominoes, to play.



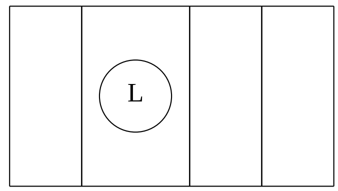
**23** **1999** A farm consists of $n$ fields in a chain. Show that Col played on this farm is always a loss for the first player if $n \geqslant 2$, and that Snort played on this farm is always a win for the first player.

**24** **1991** Evaluate the following position from a game of Col (where you may not play adjacently to a region of the same colour). Blue has played in the region marked $B$.



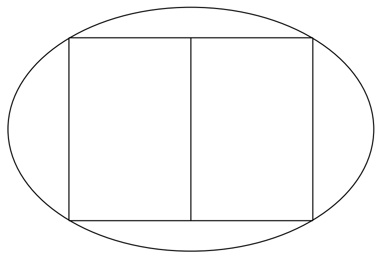
Evaluate the same position as a position in the game of Snort (where you may not play adjacently to a region of the opposite colour).

**25** **1990+2006†** Evaluate the position shown (in which the circular region has been coloured bLue by Left) in the game of Col.




Show that the same position has value $\{\{2|1\} \mid -1*\}$ in Snort.

**26** **1996** Evaluate the following position as a position in ($a$) Col [in which adjacent fields must not be the same colour]; and ($b$) Snort [in which adjacent fields must not be of opposite colours].



**27** **1992+1988**

($a$) In Red-Blue Hackenbush (in which Left chops bLue edges and Right chops Red), let $S_n$ be the stalk consisting of a blue edge at the bottom, connected to the ground, surmounted by a chain of $n$ red edges. Show that the game $S_{n+1}+S_{n+1}-S_n$ is a second player win, and deduce that $S_n$ has value $2^{-n}$. [You may assume that the value is a number.]

($b$) Deduce that $S_\infty$ has a value $\varepsilon$ satisfying $0 < \varepsilon < r$ for every positive real number $r$. Show further that $\varepsilon > \uparrow$ (where, as usual, $\uparrow = \{0 \mid *\}$).

($c$) Let **e** be an edge in a Red-Blue Hackenbush position $G$, and define the *lumber* of **e** to be **e** together with all edges of $G$ that are connected to the ground only through **e**. Let $A$ be the position obtained by removing the lumber of **e** from $G$. Show that $G-A$ is a win for the colour of **e**. Deduce that $G$ is a number.

**28** **2001** Violet (**V**) and Connie (**C**) are playing Abbreviations. In this game, an initial phrase is given, and **V** and **C** take turns in choosing a letter from a word in this phrase. **V** selects a vowel ($a$, $e$, $i$, $o$ or $u$), or **C** a consonant (all other letters), and the selected letter and all later letters in the same word are deleted. [For example, from ‘abbreviations’, **V** could move to, amongst others, ‘abbrevi’, ‘abbr’ or nothing, and **C** to ‘abbre’, ‘ab’ and so on.] **V** loses if, on her turn to move, there are no remaining vowels in the phrase, and **C** if there are no consonants.

Show that all positions in Abbreviations are numbers [REMINDER: a game is a number if all its options are numbers and every left option is less than every right option] and find this number for the word ‘banana’, assuming that Violet’s wins are positive. Show that Violet’s only winning move in

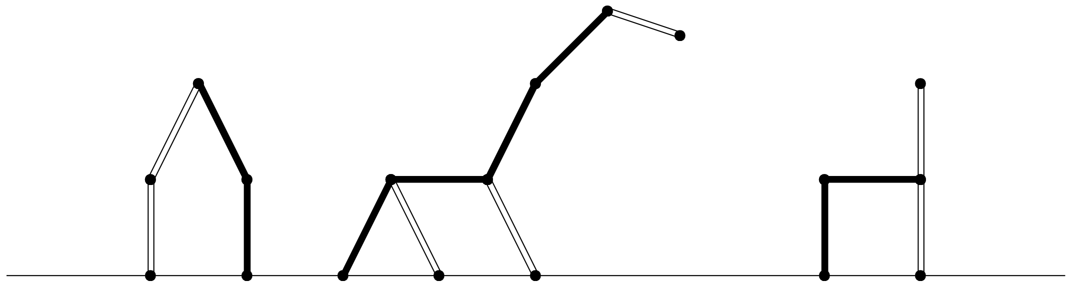
Euclid came to a new book

is ‘Euclid’ $\rightarrow$ ‘Eucl’.

The letter $y$ is sometimes a vowel [as in ‘cry’] and sometimes a consonant [as in ‘yellow’]. Suppose that any $y$ may be selected by either **V** or **C**. Show that any phrase which includes a $y$ is not a number. Determine the outcome, with best play, of the phrase

Boyish tycoons are ace

**29** **1995** Childish Hackenbush is played in the same way as Hackenbush (with, in the picture, Left chopping white edges and Right chopping black), except that no parts of the picture are allowed to drop off; that is, no edge may be chopped unless any edges to which it is connected would still be connected to the ground. Show that the house has value * and the giraffe has value $\downarrow$; and find the value of the chair.

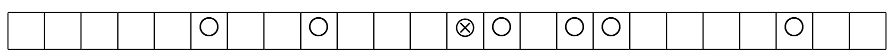


**30** **2006†** Three-player Nim is played by the usual rules of Nim; the three players play in rotation, and the game is won by the player last able to move. Show that if the starting position consists of two heaps each of size 2, then a coalition of any two players can defeat the third. [That is, that they can ensure that the winner is a member of the coalition.]

Hence or otherwise show that any position in which there are at least two heaps of size at least two can also be won by the coalition.

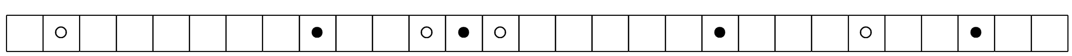
**31** **1994**

($a$) In a version of the Silver-Dollar game, you move by sliding a coin leftwards on a strip of squares, but not onto or over another coin, with the usual stopping conditions. Explain how this game relates to Nim, and find a good move in the position shown:



Suppose now that the marked coin is glued to the strip and cannot be moved. Again find a good move in the position shown.

($b$) In a version of Northcott's game, equal numbers of black and white pawns are placed alternately on a strip of squares. Black and White alternately slide a pawn of their own colour in either direction, but not onto or over another pawn, with the usual stopping conditions. Explain how this game relates to Nim, and find a good move for White in the position shown:



What happens if the condition that the pawns have alternating colours is not imposed?

**32** **1993+2005†** Cram is played with dominoes and a squared board; each domino covers exactly two squares, and a move is to place a domino so as to cover two empty squares, with the usual stopping rule. [There is no restriction, as in Domineering, that each player has to place the domino in a particular direction.] Find the sizes of the equivalent Nim heaps for ($a$) a 1×10 strip of squares; and ($b$) a 3×3 square.

**33** **1989** In Grundy's game, a move is to split a heap of matches into two smaller *unequal* heaps, with the usual stopping conventions. The following table shows the Grundy numbers, $G(n)$, for heaps of various sizes, $n$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G(n)$ | 0 | 0 | 1 | 0 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 1 |

Verify that $G(13) = 3$, and find $G(14)$. What should you play, faced with a single heap of size 13?

Given a single heap of size 27, your opponent splits it into $13+14$. What should you now play? What *should* your opponent have played?

**34** **1997** The game of Putnam is played with an assortment of sweets. You and your opponent take turns to eat the sweets, and the loser is the first person to be unable to do so. At your turn, you may eat any one sweet, provided that at least two sweets of the same kind are left; alternatively, if exactly two or three sweets of the same kind are left, you may eat both or all three of them. Construct the Grundy sequence for a heap of identical sweets, and hence or otherwise describe what your strategy should be when faced with three aniseed balls, four bon-bons, five comfits and six dragees.

**35** **1999** In Polite Nim, you may not remove more than half the matches from any pile, but the rules are otherwise as for standard Nim. Find the Grundy sequence for Polite Nim piles of up to 14 matches, and hence or otherwise find a winning move when faced with three piles, of 12, 13 and 14 matches. [As a check on your working, you will find that $G_{2n+1} = G_n$, and that there is an obvious pattern for $G_{2n}$.]

**36** **2000** Candle is played as Nim except that the heaps are arranged in a line and matches may be removed only from the end heaps. [So that as the outer heaps are reduced to zero, the 'candle burns at both ends' to expose inner heaps.]

(*a*) Show that the Candle position with three non-zero heaps, of sizes $p$, $q$ and $r$ [in that order], is a loss for the player to move if and only if $p = r \neq q$. [HINT: You are advised to construct strategies rather than Grundy numbers.]

(*b*) Find necessary amd sufficient conditions on $p$, $q$ and $r$ for the Candle position with four non-zero heaps, of sizes 1, $p$, $q$ and $r$ [in that order], to be a loss for the player to move. [You may find it helpful to consider first the case $r = 1$.]

(*c*) Find a winning move in the Candle position with heaps of sizes 2, 3, 4 and 5 [in that order].

**37** **2002** *Turning Turtles* is played with a row of coins, or 'turtles'. Two players take turns to select a coin showing heads, and to turn it over to show tails; in addition, they may, but are not required to, select a second coin to the left of the first, and turn that over as well. The second coin need not initially show heads. Eventually, all of the 'turtles' will be 'upside down', showing tails, no move will be possible, and the player who thus cannot move loses. For example, in the position shown in the figure, the rightmost coin may be flipped either by itself or with any other coin.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| H | T | T | T | T | H | H | H | T | H |

Show, by induction on the number of coins or otherwise, that the Grundy number of any position is obtained by Nim addition of the values of the coins, where the value of the $k$-th coin from the left (starting at 1) is $k$ if the coin shows heads and is 0 if it shows tails. Deduce that the position shown is worth *2, and find all winning moves in it.

**38** **1992** The game Bond is the octal game **·007**. Explain *briefly* the legal moves in Bond. Given the Grundy sequence

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $G(n)$ | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 0 | 3 | 3 | ? |

for Bond, find $G(11)$, and hence or otherwise suggest a good move in the position with heaps (or rows) of sizes 8, 9, 10 and 11.

Treblecross is played on a $1 \times n$ strip of squares. Each player in turn writes an $X$ in an empty square, and the first player to complete a line of three consecutive $X$s wins. [Since it is 'suicide' to play adjacently to or next-but-one to an existing $X$, an equivalent game is to write an $X$ on the strip and then to cross off also its immediate neighbours, with the usual stopping condition.] Find a good move in the Treblecross position

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | | | | | | | X | | | | | | | | | | | | X | |

[HINT: You may find the Grundy sequence for Bond useful.]

**39** The octal game **0.6** has the Grundy sequence given in the following table.

| *Heap size* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *Grundy number* | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 1 | 2 | 3 | ... |

Verify that a heap of size 11 has Grundy number 4.

The game of Officers is played with squads of soldiers. A move is to select a squad, and promote one of its men to be an officer; the new officer is given his own squad, which will be a non-empty part (possibly all) of the squad he was promoted from. Two rival generals play moves alternately, with the normal stopping rules. Initially the army has 11 men; how big a squad should the first officer be given?

Fish Kayles is played as Kayles, except that the ball is too big to knock down only single skittles from the middle of a row. (So the moves are to take two adjacent skittles from anywhere, or a single skittle from the end of a row.) Express Fish Kayles as an octal game, and explain how this game is related to **0.6**. What should you do, faced with a (single) row of 9 skittles?

**40** **2006** In the game Slider, one or more counters are placed on the squares of an 8×8 board. The game is played with two players and is in normal form; a move is to slide one counter any number of squares to the left, or down, or diagonally left and down, so that the game is lost by the player who finds all the counters in the bottom-left corner. It is allowed to have two or more counters on the same square.

Find the Grundy number corresponding to each square on the board. [CHECK: your numbers along the diagonal from bottom left to top right should be 0, 2, 1, 6, 7, 8, 3 and 5; and your numbers should be symmetric about this diagonal.]

Suppose that initially there are two counters, one placed six squares along the diagonal [so in the position whose value is 8], and the other placed in the top row in the square next to the top-right corner. Find a good move in this position and show that if a third counter is added to this initial position, the resulting position is still a win for the first player no matter where the third counter is added.

**41** **1988+1998** Explain the notation used to describe octal games and the rules [which need not be proved] governing the mutation of digits in this notation in forming related games. Illustrate your account by describing the games **·45** and **·177** in terms of rules governing the taking of matches from heaps and by stating the relationship between these two games. Write down the moves in **·45** from a heap of 7 matches and show how these relate to moves in **·177**.

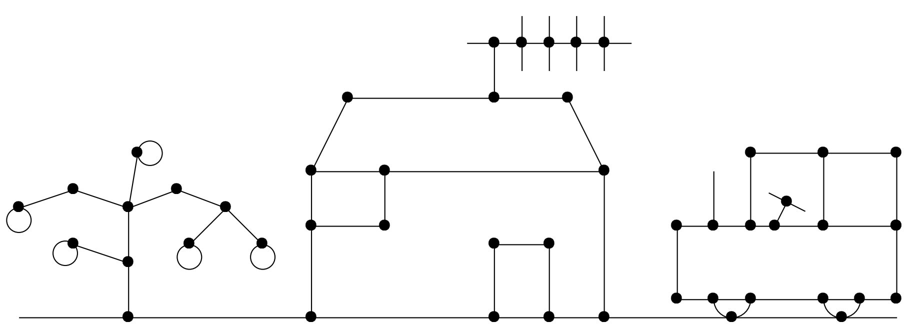
Given the following table of Grundy numbers for heaps in **·177**, verify that $G(11) = 2$. Find good moves in both **·45** and **·177** from the position with four heaps, of sizes 8, 9, 10 and 11.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $G(n)$ | 0 | 1 | 1 | 2 | 2 | 3 | 1 | 1 | 4 | 4 | 3 |

**42** **1995+2003** Explain the legal moves in the octal games **·04**, **·042** and **·0421**, using a Kayles row of length 10 as illustration. How are the Grundy sequences of these games and of **·007** related?

Construct the Grundy sequence of **·04** as far as the row of length 10, and hence or otherwise find a good move in the position with rows of lengths 8, 9 and 10.

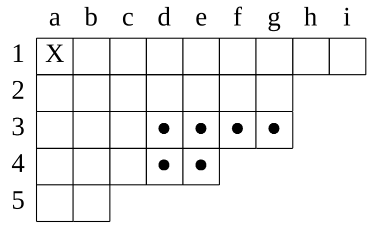
**43** ‡ Evaluate each component in the (impartial) Hackenbush position shown and suggest a good edge to cut for the player to move. Show your working, but use any general results without proof.

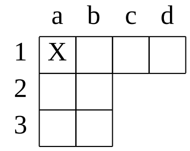


**44** **1990** The game of Chomp is played on a bar of chocolate divided into squares. A move consists of choosing and eating one of the squares, together with all squares below or to the right of the chosen square. For example, in the first position shown, choosing d3 would result in the removal of the dotted squares. Square a1 (crossed) is poisoned; whichever of the two players is forced to eat it has lost.

Show that the square a3 should be chosen if and only if row 2 is one square shorter than row 1; and that the square b2 should be chosen if and only if row 1 and column a have the same length. Analyse the second position shown.

Show that all rectangles larger than 1×1 are won for the player to move. [HINT: Consider possible replies to choosing the bottom rightmost square.] Find a winning move in the case of the 4×3 rectangle.



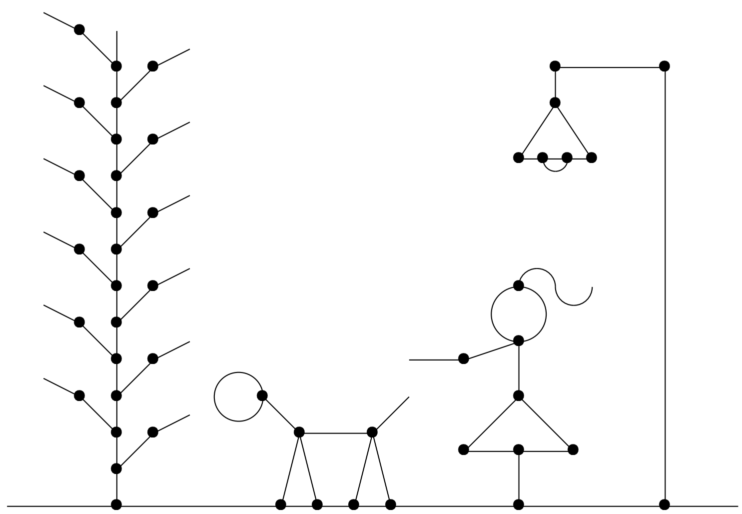




**45** (*a*) Show that any impartial Hackenbush position which includes a single sufficiently tall stalk is a win for the player to move.

(*b*) Show that a Hackenbush position which is impartial except for a single blue edge is a win for Blue if he is to move. [HINT: Consider separately those edges which would survive if the blue edge were chopped, and those which depend on the blue edge.]

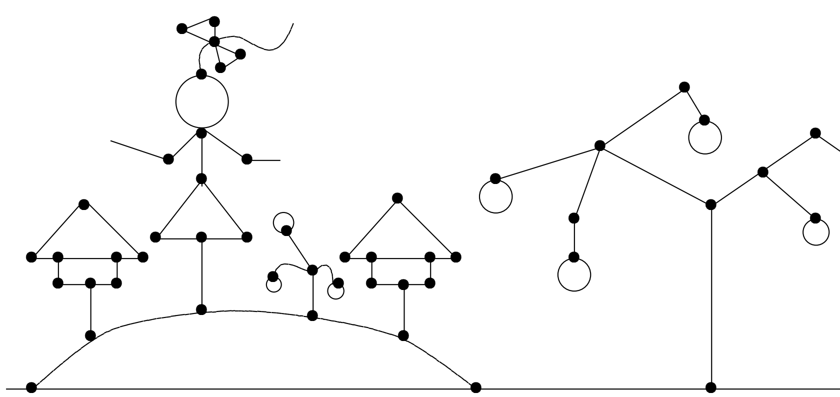
(*c*) Show that there are Hackenbush positions which are impartial except for arbitrarily many blue edges and nevertheless won for Red if he is to move.

**46** ‡ Evaluate the following picture in impartial Hackenbush, and find a good edge to chop. [Show your working, but you may use any general theorems without proof.]



**47** **2000** State the Tree Principle and the Fusion Principle for impartial Hackenbush, and prove the Tree Principle.

Evaluate the position shown, and find a winning edge to chop.

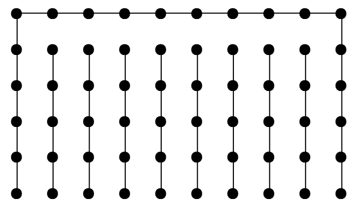


**48** **1990** The octal game **·07** is known as Dawson's Kayles. Explain the legal moves in Dawson's Kayles, considered as a variant of Kayles, showing how they relate to the octal notation.

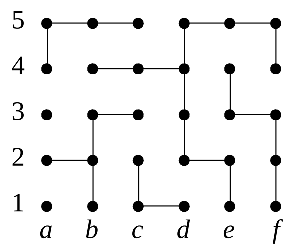
Extend the following table of Grundy numbers for Dawson's Kayles to rows of lengths 9 and 10, and hence, or otherwise, find a good move in the position with three rows, of lengths 8, 9 and 10.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| LENGTH OF ROW | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| GRUNDY NUMBER | 0 | 1 | 1 | 2 | 0 | 3 | 1 | 1 | ? | ? |

What is a Dawson's Vine in the game of Boxes? Explain (without proof) the relationship between Dawson's Vines and Dawson's Kayles. Hence, or otherwise, find a winning move in the position shown.



**49** **1989** Consider the Boxes position shown.




($a$) Explain why $c34$ is a winning move. [The notation means: draw the (vertical) line joining the dots in column $c$ and rows 3 and 4. Similarly, the two missing edges of the fourth box in the bottom row could be described as $de1$ and $d12$.] [You may use without proof the maxim that the first player should usually try to make the number of long chains have the same parity as the number of dots.] What should the final score be?

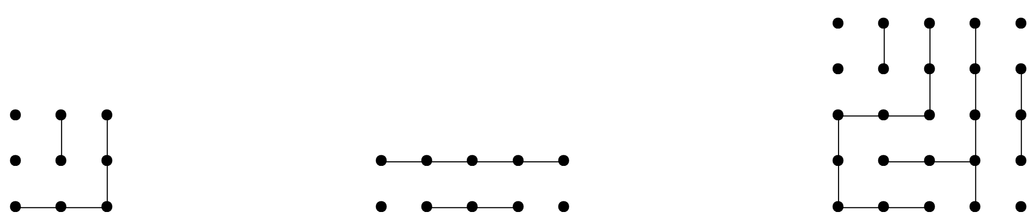
($b$) Why would $b34$ be a blunder? What should the reply be?

($c$) Is $a34$ a winning move or a blunder? Justify your answer.

**50** **1992+2006** Explain the theory underlying the taking of boxes in the impartial version of Boxes. Explain in particular the concept of *suicide* positions, and why such positions do not contradict the Sprague–Grundy theory whereby a typical Boxes position may be represented as a disjunctive sum of Nim heaps.

A Boxes position consists of $n$ disjoint chains each of length 3, all other boxes having been taken. Show that in a (partisan) game, the player to move should lose the chains by two boxes if $n$ is even, and by one box if $n$ is odd and at least 3.

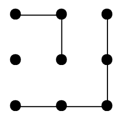
**51** **1994+2004** Show that the Boxes position *(a)* has value *3, and find the value of the position *(b)*. Hence or otherwise find a good move in the position *(c)*.



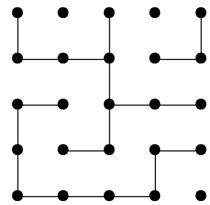
*(a)* *(b)* *(c)*

**52** **1996** Explain how the game of Boxes as normally played, in which the winner is the player who captures the majority of the boxes, is related to the impartial version of the game, in which the loser is the first player to be unable to move. State, without proof, the result relating the winner of the impartial game to the number of long chains and the number of dots.

Show that the Boxes position *(a)* has value *2. Hence or otherwise find a good move in position *(b)*. Suppose that, in the latter position, your opponent (foolishly) completes a third side of the bottom right-hand square. What should you play in your turn? Show that these moves confirm the previously-stated result.

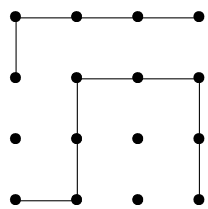


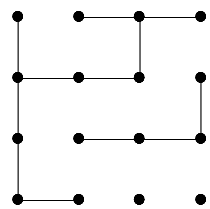
*(a)*



*(b)*

**53** **2002** Show that the impartial version of Boxes, in which the player who is first unable to play loses, is won by the first player if and only if the number of dots and the number of double-cross moves have the same parity. Explain how this rule may help in playing the usual partizan version of the game, in which the winner is the player who takes most boxes. Hence or otherwise, find good moves in each of the two positions shown in the figure.





**54** **1998** A game of Boxes played on a board with $B$ boxes [where $B > 3$] has developed $C$ long chains, $L$ loops, and there are $N$ boxes not yet in either long chains or loops; no boxes have yet been captured. Show that the outcome of the usual partisan game must, with best play, be the same as that of the impartial game if

$$C + 2L < \tfrac{1}{4}B - \tfrac{1}{2}N + 1.$$

Construct a position with no boxes captured in which the impartial and partisan games have different outcomes with best play.

**55** Explain what is meant by a *resource count* in Solitaire. Using the resource count

| | *a* | *b* | *c* | *d* | *e* | *f* | *g* |
|---|---|---|---|---|---|---|---|
| *7* | | | 0 | 8 | 0 | | |
| *6* | | | 0 | 5 | 0 | | |
| *5* | -3 | 3 | 0 | 3 | 0 | 3 | -3 |
| *4* | 2 | 2 | 0 | 2 | 0 | 2 | 2 |
| *3* | -1 | 1 | 0 | 1 | 0 | 1 | -1 |
| *2* | | | 0 | 1 | 0 | | |
| *1* | | | 0 | 0 | 0 | | |

show that the *d5, d6, d7* complement problem (in which initially the board is full except for those squares and finally it is empty except for those squares) is insoluble.

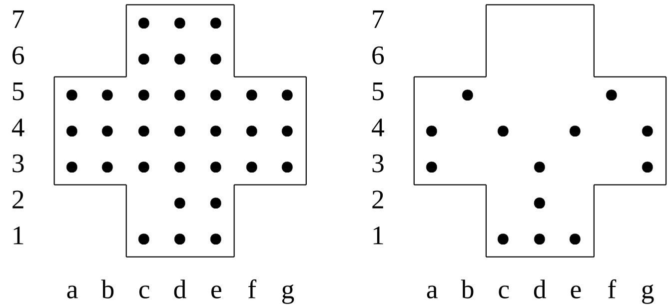
Explain why, with the same resource count, the *d7* complement problem has an effective slack of only 4. Suppose that, in a solution to this problem, the *f-g* wing is cleared by a 6-package; which of the following moves are ruled out from appearing in such a solution by this resource count: *c2–e2*, *d1–d3, d2–d4, a3–a5*, *a3–c3*, *a5–a3* and *b3–d3*?

What is the effective slack in the *d6*, *d7* complement problem? Show that in any solution to this problem, the moves *d4–d6* (twice) and *d5–d7* (once) must appear and that no other move involving *d6* or *d7* is possible. Show further that *a4* and *g4* must be cleared by the moves *a5–a3* and *g5–g3* respectively.

**56** **1991**

(*a*) Solve the Solitaire *c2*-complement problem, to reduce the single-vacancy position shown at (a) to its complement. [You need not write out the solution in full, provided that any packs used are clearly marked with the order in which they are to be applied.]

(*b*) Find a single-vacancy position which can be reduced to the 'Weeping Willow' position shown at (b) and show how to effect this reduction.



(*a*) (*b*)

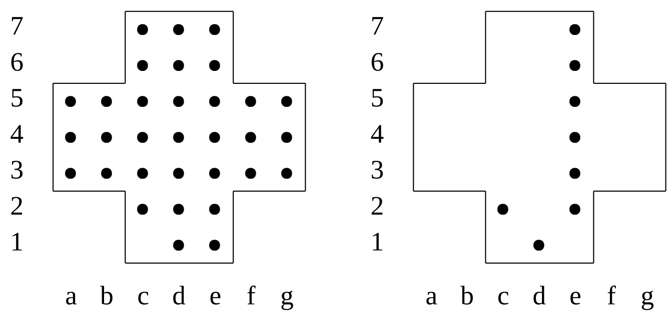
**57** Show that with the usual labelling (columns labelled *a* to *g*, rows labelled *1* to *7*), the Solitaire problem of reducing the usual starting position (only hole *d4* unoccupied) to *two* pegs of which one is in *c7* is soluble only if the other is in one of *b4*, *e1*, *e4* or *e7*.

Find a solution to the problem of reducing the usual starting position to that with just *c7* and *e7* occupied. [HINT: Use 3-packages and 6-packages, starting as in the normal solution, to reduce to the position *a3*, *a4*, *a5*, *b3*, *b4*, *b5*, *c4* and *e7*, then proceed with care.]

**58** **1993+2005**

(*a*) Solve the Solitaire *c1*-complement problem, to reduce the single-vacancy position shown at (a) to its complement. [You need not write out the solution in full, provided that any packs used are clearly marked with the order in which they are to be applied.]

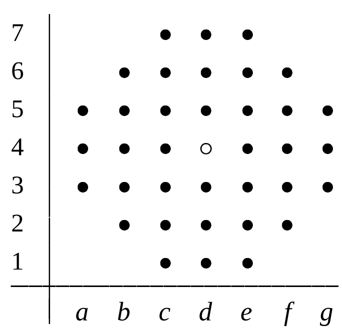
(*b*) Show that no single-vacancy Solitaire position can be reduced to the 'Letter J' position shown at (b).



(*a*) (*b*)

**59** **1988** Briefly explain the Reiss classification of Solitaire positions. Show that no complementary problem is solvable on an $m \times n$ rectangular board unless $mn$ is divisible by 3.

Show that the standard starting position [with only the centre hole unoccupied] on the 'continental' board, as shown, cannot be reduced to a single peg. In a two-peg solution with one peg on $d6$, what holes might the other final peg occupy? Find such a solution.

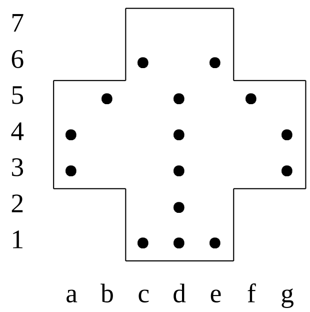


**60** **2001** Show that on a $6 \times 6$ square Solitaire board, any single-vacancy position in which the vacancy is on the edge may be reduced to its complement. You may use the standard packs freely, as long as you make clear the order in which they are to be used. [HINT:: In the case of the $a2$ complement problem, one possible start is to play $a4-a2$ and $c3-a3$ to create a 6-pack in the corner.]

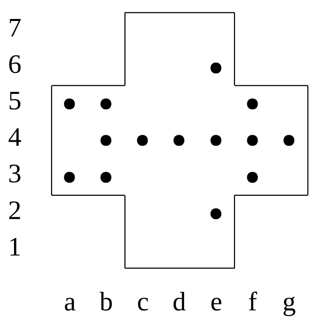
**61** **1995+2003** Find a single-vacancy Solitaire position from which the 'Tree' position shown can be reached, and describe the required moves to reach 'Tree' from this position.

[HINT: You may find it easiest to solve first the complementary problem. There is a solution which uses an L-pack—which you first have to construct—to clear out the top row; this solution is, of necessity, asymmetric. Do not over-use 3-packs.]

Explain briefly why there is no single-vacancy start position from which it is possible to reach the 'Tree' position with an extra peg at $d6$.




**62** **1999** Show that the 'Arrow' position shown [left] cannot be reached from any single-vacancy Solitaire position. [HINT: One way is to show first that the vacancy would have to be in one of $b4$, $e1$, $e4$, $e7$, then use the resource count shown right to show that none of these is suitable.]




| | | | | | | |
|---|---|---|---|---|---|---|
| | | −1 | 0 | −1 | | |
| | | 1 | 1 | 1 | | |
| −1 | 1 | 0 | 1 | 0 | 1 | −1 |
| 0 | 1 | 1 | 2 | 1 | 1 | 0 |
| −1 | 1 | 0 | 1 | 0 | 1 | −1 |
| | | 1 | 1 | 1 | | |
| | | −1 | 0 | −1 | | |

**63** **1997** The holes on a large rectangular Solitaire board have integer co-ordinates $(i, j)$. Initially, all the holes with $j > 0$ are empty.

($a$) Show that the hole $(0, 5)$ can never become occupied.

($b$) Construct an initial configuration such that the hole $(0, 4)$ can become occupied.

($c$) Show that any initial configuration such that the hole $(0, 3)$ can become occupied must contain at least 8 occupied holes. [HINT: For a suitable resource count, show that no 7 holes have sufficient resource.]

**64** Three divisions of an army defend a town, which has two roads approaching it. Four divisions attack the town along the roads. (Divisions are indivisible.) A defending division is equivalent to two attacking divisions. A stronger force always beats a weaker force; if the attackers and defenders are equivalent, then each wins with probability ½. Write down the possible strategies for the two armies, and construct the payoff matrix for the attackers' chances of entering the town. What are the upper and lower values of this 'game'?

Find the value of the game, and the optimal strategies.

**65** **1989+2003** A dog chases a rabbit. The rabbit escapes into a tunnel which has three exit holes, $A$, $B$ and $C$. $B$ and $C$ are relatively close together, but some distance from $A$. The rabbit, being hungry, must emerge from one of the holes. The dog can lie in wait near a hole, in which case he catches the rabbit if and only if it emerges from this hole. Another strategy is to wait between holes $B$ and $C$, in which case the rabbit escapes if it emerges from $A$, but is caught with probability $p$ if it emerges from $B$ or $C$. Finally the dog can wait at a vantage point overlooking all three holes, in which case he catches the rabbit with probability $q$, no matter which hole it emerges from.

Express this scenario as a matrix game, and show that these latter two strategies are dominated if $p < \frac{1}{2}$ or $q < \frac{1}{3}$, respectively.

Find the best strategies for dog and rabbit in the case $p = \frac{3}{4}$, $q = \frac{2}{5}$.

**66** **1990** Left and Right play a game in which each independently stakes either £1 or £2. If the total amount staked is an even number of pounds, then Left takes all the staked money; if it is odd, then Right takes the money. Analyse this game, showing that it is fair, that Left should stake £1 with probability $\frac{2}{3}$ and that Right should stake £1 with probability $\frac{1}{2}$.

Suppose now that the number of pounds staked is not necessarily 1 or 2, but that Left must select from a set $\mathscr{L}$ of positive integers, and Right must similarly select from a set $\mathscr{R}$. Show that Left and Right should select their stakes from at most two different members of their respective sets. Analyse the cases

($a$) $\mathscr{L} = \mathscr{R} = \{1, 2, 3, 4, 5, 6\}$;

($b$) $\mathscr{L} = \mathscr{R} = \{2, 4, 6, 8, 9, 13\}$;

($c$) $\mathscr{L} = \{1, 2, 3, 4, 5, 6\}$, $\mathscr{R} = \{2, 4\}$.

**67** **1998** Left and Right independently show one, two or three fingers. If the total number of fingers on display is odd, then Left wins; if even, then Right wins. The loser pays the winner £1 for each finger on display [so between £2 and £6].

Analyse this game, determining the optimal strategy for each player and the expected outcome.

**68** **1991** Find, for all values of $\alpha$, the value of the two-person zero-sum game with payoff matrix (for Left)

| | | Right 1 | Right 2 |
|---|---|---|---|
| Left | 1 | 1 | 0 |
| | 2 | $\alpha$ | 1 |

Show that if $\alpha \geqslant 1$ each player should play strategy 2. What should they do if $\alpha < 1$?

In World War II, a favourite tactic for fighter pilots was to dive on opponents 'out of the sun'. If you were unseen, you certainly survived [and had a good chance of shooting down one or more opponents]; even if you were seen, there was a good chance, say $\frac{19}{20}$, that you would survive because of your speed in the dive. This tactic led to defending pilots wearing sunglasses and staring at the sun. This in turn led to a new attacking tactic of climbing slowly towards your opponents, hoping he was staring at the sun. Your slow climbing speed made you a sitting duck however, if you were spotted.

Formulate this situation as a matrix game, with payoff the survival chances of the attacking pilot. Show that the game has value $\frac{20}{21}$, and that the attacking pilot should play 'sitting duck' one time in twenty-one.

What is the value if the pilot refuses to play sitting duck? Comment on the realism of your findings.

**69** **1992**

Tweedledum and Tweedledee
Agreed to have a battle;
For Tweedledum said Tweedledee
Had spoiled his nice new rattle.

The fight was custard pies at 20 paces. Each twin has one pie. Dum has first go, and can either throw his pie or pass. If he hits (probability $\frac{1}{3}$), the fight is over. If he misses or passes, Dee has the same choice. If the fight goes on, they advance 10 paces and try again, this time with probability $\frac{3}{4}$ of hitting. If the fight is still not resolved, they advance another 10 paces, and at point blank range a hit is certain. What should their strategies be, and what is the probability that Dum will win?

Dum and Dee have now grown up, slightly, and fight with phaser stun guns instead of custard pies. The gun will fire only once per fight, but you cannot observe whether or not the gun has been fired (unless you are hit). What now are the best strategies? [Assume that guns are no more and no less accurate than pies, and that the fight is drawn if both miss.]

**70** **1993+2004** In the game 'Show Me a Picture', Left draws a card at random from a shuffled standard pack of cards, looks at it, and then chooses whether to bet £1 or £5. Right does not see the card, and either concedes (paying Left the amount bet) or doubles. If Right doubles, Left shows the card, winning the bet (so receiving £2 or £10) if the card is a picture (King, Queen or Jack), and losing otherwise.

What pure strategies do Left and Right have? Express the game in matrix form, and show that Left should always bet £5 on a picture. Find the optimal strategies for each player, and the value of the game.

**71** **1994+2005** Len and Rick are trying to determine who will buy the next round of drinks. L has caught a fly, and also has a realistic imitation fly; R has a fly-swat. L places on the table either the real fly or the imitation, covered by his hand; L removes his hand, and as he does so R can either swat the fly or pass. If R swats the real fly, L buys the drinks; if he swats the imitation, R buys the drinks. If R passes on the real fly, the fly escapes, the game is over, and L and R share the round. If R passes on the imitation, then they try once more; but if the second round is also inconclusive (that is, R again passes on the imitation), then L buys the round rather than try a third time.

Write down the three available pure strategies for each of L and R. Formulate this scenario as a matrix game, and solve it to find the optimal (mixed) strategies for L and R.

**72** **1996** The evil Sir Mordred has imprisoned Rosamund and her three sisters (total value 4) in the Bower, and Lily and her two sisters (total value 3) in the Valley. The gallant Sir Galahad sets out to rescue either Rosamund or Lily; independently, Sir Mordred sets out to foil the rescue. If they set out to the same place, Sir Mordred has an evens chance of arriving first, in which case Sir Galahad returns empty-handed. Otherwise, Sir Galahad effects his rescue. Describe this scenario as a matrix game, determine the optimal strategies for each player, and show that the game has value $2\frac{4}{7}$ to Sir Galahad.

One of Rosamund's sisters has now escaped, so the reward for rescuing Rosamund is only 3. What difference should this make to the strategies and to the value of the game in the cases

($a$) neither Sir Galahad nor Sir Mordred has heard this news;
($b$) Sir Galahad has heard, and is sure that Sir Mordred hasn't;
($c$) Sir Mordred has heard, and is sure that Sir Galahad hasn't; and
($d$) both knights have heard?

**73** **1997** You control two companies: one writes Chess programs, the other writes Bridge programs. After the first year of trading, C should have a tax bill of £3 million, and B should have a tax bill of £1 million. However, by 'creative' accounting, you can make it appear, for either or both companies, as though you should not have to pay any tax at all. The authorities only have the resources to investigate one company; if the investigated company has mis-behaved, the 'creativity' will certainly be discovered, and it will have to pay all the tax due plus a penalty of $p$% of the tax due. Describe this scenario as a matrix game, stating clearly what the available pure strategies are.

($a$) In the case $p = 50$ [so you pay 'tax-and-a-half'], what strategy should you adopt to minimise your expected tax bill?

($b$) For what values of $p$ is complete honesty a worthwhile policy [in terms merely of minimising your tax bill for the one year!]?

**74** **1999+2006** Len tosses a fair coin, keeping it covered. He looks at the coin, and says 'Heads' or 'Tails', not necessarily truthfully. Rick can either believe Len, and pay him £1 for 'Heads' or nothing for 'Tails', or can call his bluff, in which case Rick pays Len £2 if Len told the truth and Len pays Rick £2 if he lied.

Formulate this game as a matrix game, stating clearly what the available pure strategies are, and showing clearly how a typical matrix element is calculated. Show that Len should never lie when the coin shows 'Heads' and that Rick should always believe a call of 'Tails', and hence or otherwise determine the best strategy for each player.

**75** **1988+1995** Briefly explain the concept of an *evolutionary stable strategy* (E.S.S.) in an evolutionary game. A population of male deer fight over females. The males have two strategies:

*Hawk:* fight until the opponent runs away or one contestant is injured;
*Dove:* run away rather than fight.

The reward for winning is $W$ and the penalty for being injured is $I$. If two males fight while playing the same strategy, each wins with probability $\frac{1}{2}$. Show that if a male playing Hawk with probability $y$ fights one playing Hawk with probability $x$, the expected payoff for the first male is

$$\tfrac{1}{2}W\left(1-x+y-\frac{I}{W}xy\right).$$

Deduce that if $I < W$, then the pure Hawk strategy, $x = 1$, is E.S.S, but if $I > W$, then neither pure Hawk nor pure Dove is E.S.S.

A third possible male strategy is

*Bully:* make a show of fighting in case the opponent runs away, but run away before being injured.

Show that Bullies dominate Doves, in the sense that no E.S.S. can contain Doves, and deduce that if $I > W$ the E.S.S., if one exists, must be to play Hawk with probability $W/I$ and otherwise Bully.

A fourth possible male strategy is

*Retaliate:* play Dove unless attacked (by a Hawk), in which case play Hawk.

Show that if $I > W$, no **ESS** can contain Retaliators, and that if $I < W$, no **ESS** can contain Doves or Hawks. If $I < W$, is pure Retaliator an **ESS**?

**76** **2000** Two peacocks are ‘displaying’ for possession of a lawn; whichever displays the longer will gain the lawn, of value $V$. There is a penalty of $w(t)$ for displaying for a time $t$. Explain briefly why no pure strategy can be ESS, provided that $w$ is continuous.

Show that a peacock that is prepared to display for a time $p$ has an expected payoff of

$$\int_0^p (V - w(q))\, f(q)\,\mathrm{d}q + \int_p^\infty -w(p)\, f(q)\,\mathrm{d}q$$

when playing against a peacock that selects his maximum display time at random from a distribution with pdf $f$. Deduce that if this selection is ESS, then $f$ must satisfy

$$V f(p) = w'(p)\int_p^\infty f(q)\,\mathrm{d}q.$$

Find a candidate ESS in the case where $w(p) = p^2$.

**77** **2001** Cain and Abel fight a duel. They each have a spear and are initally far apart. They walk towards each other. When the separation is $x$, Cain’s chance of killing Abel by throwing his spear is $c(x)$, and similarly Abel’s chance is $a(x)$, where you may assume that $a$ and $c$ are continuous and monotonic, that $c(0) = a(0) = 1$ and that $c(x_0) = 0$ where $x_0$ is the initial separation.

Assuming that these functions are known to both players, show that the optimal strategy for each player is to wait until the separation is such that $a(x) + c(x) = 1$ and then to throw, unless the opponent has already thrown and missed.

Consider the two cases in which Abel is known to be *(a)* optimistic; and *(b)* pessimistic about his own skill. Thus Abel believes his chance to be $a'(x)$, where in case *(a)* $a'(x) \geqslant a(x)$, for all $x$, and in case *(b)* $a'(x) \leqslant a(x)$. How, if at all, should Cain modify his strategy in each case?

*End*