Question 13 Solution

a)
$$x + 2y - z = 2$$
$$x + y + z = 3$$
$$3x + 2y + z = 1$$

In matrix form the system is $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{Y}$ where

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad \underset{\sim}{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \underset{\sim}{Y} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

The augmented matrix for the system is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \end{array}\right)$

Using the method of row reduction:

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - R_1 \\ R_3' = R_3 - 3R_1 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & -4 & 4 & -5 \end{array}\right)$$

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_3 - 4R_2 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -4 & -9 \end{array}\right)$$

So
$$-4z = -9 \rightarrow z = \frac{9}{4}$$
$$-y + 2z = 1 \rightarrow y = \frac{9}{2} - 1 = \frac{7}{2}$$
$$x + 2y - z = 2 \rightarrow x = 2 + \frac{9}{4} - 7 = -\frac{11}{4}$$

Hence the solution is

$$x = \frac{-11}{4}, \quad y = \frac{7}{2}, \quad z = \frac{9}{4}$$

13b

$$x + y + 4z = 3$$
$$2x + y + 2z = 4$$
$$3x + y \qquad = 5$$

In matrix form the system is $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{Y}$ where

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \\ 3 & 1 & 0 \end{pmatrix}, \quad \underset{\sim}{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \underset{\sim}{Y} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

The augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 2 & 1 & 2 & 4 \\ 3 & 1 & 0 & 5 \end{array}\right)$

Using the method of row reduction:

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & -1 & -6 & -2 \\ 0 & -2 & -12 & -4 \end{array}\right)$$

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_3 - 2R_2 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & -1 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

One of the equations is redundant, therefore there will be an infinite set of solutions.

$$-y - 6z = -2 \quad \rightarrow \quad y = 2 - 6z$$
$$x + y + 4z = 3 \quad \rightarrow \quad x = 3 - (2 - 6z) - 4z = 1 + 2z$$

Hence the set of solutions is

$$x = 1 + 2t, \quad y = 2 - 6t, \quad z = t \qquad \text{for } -\infty < t < \infty$$

Question 14 Solution

The eigenvalues of the Matrix $\underset{\sim}{A}$ are the solutions of the determinantal equation $\det(\underset{\sim}{A}-\lambda \underset{\sim}{I})=0$.

So if $\underset{\sim}{A}=\begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix}$ we need to solve $\begin{vmatrix} 2-\lambda & -3 & 1 \\ -3 & 6-\lambda & -3 \\ 1 & -3 & 2-\lambda \end{vmatrix}=0$

$$\left.\begin{array}{l} R_1'=R_1-R_3 \\ R_2'=R_2 \\ R_3'=R_3 \end{array}\right\} \rightarrow \begin{vmatrix} 1-\lambda & 0 & -1+\lambda \\ -3 & 6-\lambda & -3 \\ 1 & -3 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & -1 \\ -3 & 6-\lambda & -3 \\ 1 & -3 & 2-\lambda \end{vmatrix}=0$$

$$\left.\begin{array}{l} C_1'=C_1+C_3 \\ C_2'=C_2 \\ C_3'=C_3 \end{array}\right\} \rightarrow (1-\lambda)\begin{vmatrix} 0 & 0 & -1 \\ -6 & 6-\lambda & -3 \\ 3-\lambda & -3 & 2-\lambda \end{vmatrix} = (-1)(1-\lambda)\begin{vmatrix} -6 & 6-\lambda \\ 3-\lambda & -3 \end{vmatrix}=0$$

$$= -(1-\lambda)\left[18-(3-\lambda)(6-\lambda)\right] = -(1-\lambda)\left[-\lambda^2+9\lambda\right]=0$$

$$= -\lambda(1-\lambda)(\lambda-9)=0$$

<u>So the eigenvalues of $\underset{\sim}{A}$ are $\lambda=0,1,9$.</u>

To find the eigenvector corresponding to eigenvalue $\lambda=\lambda_1$ we must solve the system of equations $\underset{\sim}{A}\underset{\sim}{X}=\lambda_1\underset{\sim}{X}$, which may be written $(\underset{\sim}{A}-\lambda_1\underset{\sim}{I})\underset{\sim}{X}=\underset{\sim}{0}$

When $\lambda=0$ we must therefore solve the system

$$\begin{aligned} 2x-3y+z&=0 \quad &(1)\\ -3x+6y-3z&=0 \quad &(2)\\ x-3y+2z&=0 \quad &(3) \end{aligned}$$

$3\times(1)+(2) \rightarrow \quad 3x-3y=0 \quad \rightarrow \quad y=x$

Subs. into (1) $\rightarrow \quad 2x-3(x)+z=0 \quad \rightarrow \quad z=x$

[notice: equation (3) is automatically satisfied by $y=x$, $z=x$ for <u>all</u> x] Hence the eigenvector corresponding to eigenvalue $\lambda=0$ is $\underset{\sim}{V}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $t\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a solution of the system for all t.

Question 14 solution continued

When $\lambda = 1$ we must solve the system

$$x - 3y + z = 0 \quad (4)$$
$$-3x + 5y - 3z = 0 \quad (5)$$
$$x - 3y + z = 0 \quad (6)$$

[notice that equations (4) and (6) are identical]

$3 \times (4) + (5) \rightarrow \quad -4y = 0 \rightarrow y = 0$

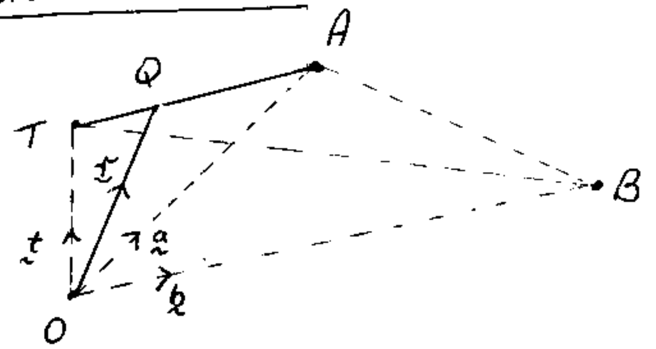
subs. into (4) $\rightarrow \quad x - 3(0) + z = 0 \Rightarrow z = -x$

Hence the eigenvector corresponding to eigenvalue $\lambda = 1$ is

$V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a solution of the system

for all t.

Question 15 solution



Let the position vectors of the points T, A and B be $\underset{\sim}{t}$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. It is given that $\underset{\sim}{t}=(2,2,0)$, $\underset{\sim}{a}=(4,3,1)$ and $\underset{\sim}{b}=(-3,1,2)$.

a) Let the position vector of Q be $\underset{\sim}{r}$ then

$$\underset{\sim}{r}=\underset{\sim}{t}+\vec{TQ}$$

where $\vec{TQ}=\frac{2}{5}\vec{TA}$.

$\vec{TA}=\underset{\sim}{a}-\underset{\sim}{t}=(4,3,1)-(2,2,0)=(2,1,1)$ so $\vec{TQ}=\frac{2}{5}(2,2,1)=\left(\frac{4}{5},\frac{2}{5},\frac{2}{5}\right)$,

and therefore $\underline{\underset{\sim}{r}=(2,2,0)+\left(\frac{4}{5},\frac{2}{5},\frac{2}{5}\right)=\left(\frac{14}{5},\frac{12}{5},\frac{2}{5}\right)}$.

b) The plane π is perpendicular to the vector $\underset{\sim}{\rho}=\vec{TA}\times\vec{TB}$

and the equation of π is $\underset{\sim}{r}\cdot\underset{\sim}{\rho}=\text{const.}$

We have already found that $\vec{TA}=(2,1,1)$,

$\vec{TB}=\underset{\sim}{b}-\underset{\sim}{t}=(-3,1,2)-(2,2,0)=(-5,-1,2)$

so $$\underset{\sim}{\rho}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & 1 & 1\\ -5 & -1 & 2\end{vmatrix}=\underset{\sim}{i}(2+1)-\underset{\sim}{j}(4+5)+\underset{\sim}{k}(-2+5)=(3,-9,3)$$

The equation of π is therefore $\underset{\sim}{r}\cdot(3,-9,3)=\text{const}=c$ (say)

ie $3x-9y+3z=c$.

the plane passes through T so $3(2)-9(2)+3(0)=c$

which gives $c=-12$. (the plane also passes through A and B so we could substitute the coordinates of these points into the equation to find c)

The equation of π is therefore $3x-9y+3z=-12$

which can be reduced to

$$\underline{x-3y+z=-4}$$

Question 15 solution continued

c) If we write the equation of π $\underset{\sim}{r} \cdot \hat{\underset{\sim}{p}} = d$ then $|d|$ is the perpendicular distance of the origin from π.

$$x - 3y + z = -4 \quad \rightarrow \underset{\sim}{r} \cdot \left(\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right) = -\frac{4}{\sqrt{11}}$$

so $\frac{4}{\sqrt{11}}$ is the perpendicular distance of O from π.

Question 16 solution

a) R_1 and R_2 will collide if there is a time t for which

$$\underset{\sim}{r}_1(t) = \underset{\sim}{r}_2(t)$$

ie $$(t^2, 2t, 5) = (4t-3, 2t^2, 5)$$

Two vectors are equal if their components are equal

ie $\underset{\sim}{i}$: $t^2 = 4t-3 \rightarrow t^2-4t+3=0 \rightarrow (t-3)(t-1)=0$

$\underset{\sim}{j}$: $2t = 2t^2 \rightarrow t=0$ or $t=1$

$\underset{\sim}{k}$: $5=5$

the $\underset{\sim}{i}$ components are equal <u>and</u> the $\underset{\sim}{j}$ components are equal only when $t=1$. Hence the rockets collide at $t=1$.

(i) $\underset{\sim}{v}_1 = \dot{\underset{\sim}{r}}_1(t) = (2t, 2, 0)$, at $t=1$ $\underset{\sim}{v}_1 = (2,2,0)$

$\underset{\sim}{v}_2 = \dot{\underset{\sim}{r}}_2(t) = (4, 4t, 0)$, at $t=1$ $\underset{\sim}{v}_2 = (4,4,0)$

hence $\underset{\sim}{v}_2 = 2\underset{\sim}{v}_1$ ie $|\underset{\sim}{v}_2| = 2|\underset{\sim}{v}_1|$ and $\hat{\underset{\sim}{v}}_2 = \hat{\underset{\sim}{v}}_1$.

(ii) $\underset{\sim}{a}_1 = \ddot{\underset{\sim}{r}}_1(t) = (2,0,0)$ for all t

$\underset{\sim}{a}_2 = \ddot{\underset{\sim}{r}}_2(t) = (0,4,0)$ for all t

$|\underset{\sim}{a}_2| = 2|\underset{\sim}{a}_1|$ and $\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 = 0$ hence $\underset{\sim}{a}_1 \perp \underset{\sim}{a}_2$.

b)
$$2x + 8y + 3z = 0 \quad (1)$$
$$x + 2y + z = 0 \quad (2)$$

$(1) - 3(2) \longrightarrow$ $-x + 2y = 0 \rightarrow y = \frac{1}{2}x$

Subs into (2) $\rightarrow$ $x + 2\left(\frac{1}{2}x\right) + z = 0 \rightarrow z = -2x$

Hence $y = \frac{1}{2}x$, $z = -2x$ satisfy the equations for all values of x. The solution is therefore the set of values of (x,y,z) on the straight line

$$x = \lambda, \quad y = \tfrac{1}{2}\lambda, \quad z = -2\lambda \quad \text{for } -\infty < \lambda < \infty$$

The line passes through the origin. The vector equation of the line is

$$\underset{\sim}{r} = \underset{\sim}{0} + \lambda\left(1, \tfrac{1}{2}, -2\right)$$

ie $$\underline{\underset{\sim}{r} = \lambda\left(1, \tfrac{1}{2}, -2\right) \text{ for } -\infty < \lambda < \infty}$$

Question 16 Solutions continued

Let the points on the line L_1 be labelled $\underset{\sim}{r}_1$ and the points on line L_2 be labelled $\underset{\sim}{r}_2$, then

$\underset{\sim}{r}_1 = \lambda\left(1, \frac{1}{2}, -2\right)$ and $\underset{\sim}{r}_2 = (1, 2, 3) + \mu\left(1, 1, -\frac{1}{3}\right)$

L_1 and L_2 intersect when $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$ ie we must find the values of λ and μ for which $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$.
Comparing components in the equation $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$ we have

$\underset{\sim}{i}$: $\lambda = 1 + \mu$ (1)

$\underset{\sim}{j}$: $\frac{1}{2}\lambda = 2 + \mu$ (2)

$\underset{\sim}{k}$: $-2\lambda = 3 - \frac{1}{3}\mu$ (3)

(1) − (2) $\rightarrow$ $\frac{1}{2}\lambda = -1$ so $\lambda = -2$

Subs into (1) $\rightarrow$ $\mu = -3$

Note: Equation (3) is automatically satisfied.

Hence the lines intersect at $(-2, -1, 4)$

(ie. $\underset{\sim}{r}_1 = -2\left(1, \frac{1}{2}, -2\right)$, $\underset{\sim}{r}_2 = (1, 2, 3) - 3\left(1, 1, -\frac{1}{3}\right)$)