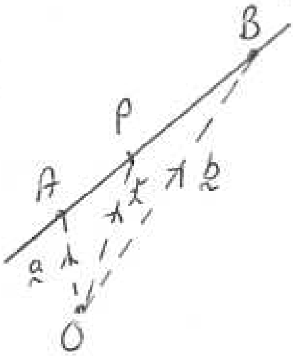
Question 13

Let $\underset{\sim}{a} = (1,2,5)$, $\underset{\sim}{b} = (2,1,7)$ and $\underset{\sim}{c} = (-3,3,6)$, the position vectors of the points A, B and C.

(a)



Let $\underset{\sim}{r}$ be the position vector of a general point on the line which passes through A and B.

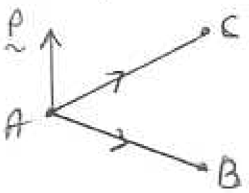
$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = (2,1,7) - (1,2,5) = (1,-1,2)$.

The equation of the line through A and B is then

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \overrightarrow{AB} = (1,2,5) + \lambda(1,-1,2)$$
for $-\infty < \lambda < \infty$.

(b) The equation of the plane P will be $\underset{\sim}{r}\cdot\underset{\sim}{p} = \text{const} = K$ where $\underset{\sim}{p}$ is a vector perpendicular to the plane P.

The plane P is perpendicular to $\overrightarrow{AB} \times \overrightarrow{AC}$. Let $\underset{\sim}{p} = \overrightarrow{AB} \times \overrightarrow{AC}$



$\overrightarrow{AB} = (1,-1,2)$ from above

$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a} = (-3,3,6) - (1,2,5) = (-4,1,1)$

so $\underset{\sim}{p} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -1 & 2 \\ -4 & 1 & 1 \end{vmatrix} = \underset{\sim}{i}(-1-2) - \underset{\sim}{j}(1+8) + \underset{\sim}{k}(1-4)$

$= (-3, -9, -3)$.

So the equation of P is

$\underset{\sim}{r}\cdot(-3,-9,-3) = K$. $\rightarrow -3x - 9y - 3z = K$

The plane passes through A so $(1,2,5)\cdot(-3,-9,-3) = K = -36$

giving the required equation $-3x - 9y - 3z = -36$

ie. $\underline{x + 3y + z = 12}$

(c) A horizontal plane has the equation $\underset{\sim}{r}\cdot(0,0,1) = \text{const} = C$.

If θ is the angle between the planes then θ is the angle between the normals to the planes. $\cos\theta = \hat{\underset{\sim}{p}}\cdot(0,0,1)$

Hence $\theta = \cos^{-1}\left(\dfrac{(1,3,1)\cdot(0,0,1)}{\sqrt{11}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{11}}\right) \sim 72.45^\circ$

Question 14

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + (\rho+1)x_3 &= 0 \\ \rho x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}\right\} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & (\rho+1) \\ \rho & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

that is $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{0}$ where $\underset{\sim}{A} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & (\rho+1) \\ \rho & 3 & 2 \end{pmatrix}$ and $\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The system has non trivial solutions when $\det \underset{\sim}{A} = 0$

That is when $\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & (\rho+1) \\ \rho & 3 & 2 \end{vmatrix} = 0$

Manipulating columns we obtain

$$\left.\begin{aligned} C_1' &= C_1 - 2C_2 \\ C_2' &= C_2 \\ C_3' &= C_3 \end{aligned}\right\} \rightarrow \Delta = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & (\rho+1) \\ \rho-6 & 3 & 2 \end{vmatrix}$$

Expanding along the first column we obtain

$$\Delta = +(\rho-6)\begin{vmatrix} 1 & 1 \\ 1 & (\rho+1) \end{vmatrix} = (\rho-6)\left[(\rho+1)-1\right] = \rho(\rho-6)$$

So $\Delta = 0$ when $\rho(\rho-6) = 0$ ie $\underline{\text{when } \rho = 0 \text{ or } \rho = 6}$

$\underline{\text{when } \rho = 0}$ the equations are

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \quad (1) \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_2 + 2x_3 &= 0 \quad (2) \end{aligned}$$

equation (2) gives $x_2 = -\frac{2}{3}x_3$, substituting into (1) gives $x_1 = -\frac{1}{6}x_3$

So for any value of x_3, say $x_3 = t$ for $-\infty < t < \infty$

the solution is $x_1 = -\frac{1}{6}t$, $x_2 = -\frac{2}{3}t$, $x_3 = t$

$\underline{\text{when } \rho = 6}$ the equations are

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \quad (1) \\ 2x_1 + x_2 + 7x_3 &= 0 \quad (2) \\ 6x_1 + 3x_2 + 2x_3 &= 0 \quad (3) \end{aligned}$$

equation (1) - equation (2) $\rightarrow$ $-6x_3 = 0 \rightarrow x_3 = 0$

Substituting into (1) $\rightarrow$ $x_2 = -2x_1$

So for any value of x_1, say $x_1 = t$ for $-\infty < t < \infty$

the solution is $\underline{x_1 = t,\ x_2 = -2t,\ x_3 = 0}$

We should check that equation (3) is satisfied

Question 15

The eigenvalues of the matrix $\underset{\sim}{A} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix}$ are the solutions of the determinantal equation $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$, that is the solutions of
$$\Delta = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 3-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

Carrying out column manipulations we obtain
$$\left.\begin{array}{l} C_1' = C_1 - C_3 \\ C_2' = C_2 \\ C_3' = C_3 \end{array}\right\} \rightarrow \Delta = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 0 & 3-\lambda & -2 \\ -(1-\lambda) & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & -2 & 0 \\ 0 & 3-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

Carrying out row manipulations we obtain
$$\left.\begin{array}{l} R_1' = R_1 + R_3 \\ R_2' = R_2 \\ R_3' = R_3 \end{array}\right\} \rightarrow \Delta = (1-\lambda)\begin{vmatrix} 0 & -4 & 1-\lambda \\ 0 & 3-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = -(1-\lambda)\begin{vmatrix} -4 & 1-\lambda \\ 3-\lambda & -2 \end{vmatrix}$$

where we have expanded along the first column.

Hence $\Delta = -(1-\lambda)[8 - (1-\lambda)(3-\lambda)] = (1-\lambda)(\lambda^2 - 4\lambda - 5) = 0$
$$= (1-\lambda)(\lambda-5)(\lambda+1) = 0,$$

which gives $\lambda = 1, -1, 5$. These are the eigenvalues.

To find the eigenvector corresponding to the eigenvalue $\lambda = 1$ we need to solve the system of equations $(\underset{\sim}{A} - \lambda \underset{\sim}{X}) = 0$ where $\underset{\sim}{X} = (x_1, x_2, x_3)^T$ and $\lambda = 1$, that is the system
$$\begin{array}{lcl} -2x_2 = 0 & \rightarrow & x_2 = 0 \\ -2x_1 + 2x_2 - 2x_3 = 0 & \rightarrow & x_3 = -x_1 \\ -2x_2 = 0 & & \end{array}$$

hence the eigenvector corresponding to eigenvalue $\lambda = 1$ is $\underset{\sim}{v} = (1, 0, -1)^T$

To find the eigenvector corresponding to eigenvalue $\lambda = -1$ we need to solve the system of equations $(\underset{\sim}{A} - \lambda \underset{\sim}{X}) = 0$ when $\lambda = -1$, that is
$$\begin{array}{lcl} 2x_1 - 2x_2 = 0 & \rightarrow & x_1 = x_2 \\ -2x_1 + 4x_2 - 2x_3 = 0 & \rightarrow & x_2 = x_3 \\ -2x_2 + 2x_3 = 0 & & \end{array} \Bigg\} \rightarrow x_1 = x_2 = x_3$$

Hence the eigenvector corresponding to eigenvalue $\lambda = -1$ is $\underset{\sim}{v} = (1, 1, 1)^T$

Question 16

The vector equations of the planes: $\underset{\sim}{r}\cdot(-1,3,-1)=7$

$$\underset{\sim}{r}\cdot(0,-1,2)=-3$$
$$\underset{\sim}{r}\cdot(2,-1,0)=1$$

the cartesian form of the equations:

$$\begin{aligned}-x+3y-z&=7\\ -y+2z&=-3\\ 2x-y\quad&=1\end{aligned}$$

which can be written in the matrix form $\underset{\sim}{A}\underset{\sim}{X}=\underset{\sim}{H}$

where $\underset{\sim}{A}=\begin{pmatrix}-1&3&-1\\0&-1&2\\2&-1&0\end{pmatrix}$, $\underset{\sim}{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\underset{\sim}{H}=\begin{pmatrix}7\\-3\\1\end{pmatrix}$

The augmented matrix of the system is $(\underset{\sim}{A}\vdots\underset{\sim}{H})=\left(\begin{array}{ccc|c}-1&3&-1&7\\0&-1&2&-3\\2&-1&0&1\end{array}\right)$.

Carrying out Row Reduction:

$$\left.\begin{aligned}R_1'&=R_1\\R_2'&=R_2\\R_3'&=R_3+2R_1\end{aligned}\right\}\rightarrow\left(\begin{array}{ccc|c}-1&3&-1&7\\0&-1&2&-3\\0&5&-2&15\end{array}\right)$$

$$\left.\begin{aligned}R_1'&=R_1\\R_2'&=R_2\\R_3'&=R_3+5R_2\end{aligned}\right\}\rightarrow\left(\begin{array}{ccc|c}-1&3&-1&7\\0&-1&2&-3\\0&0&8&0\end{array}\right)$$

Hence

$$\begin{aligned}8z=0&\rightarrow z=0\\ -y+2(0)=-3&\rightarrow y=3\\ -x+3(3)-1(0)=7&\rightarrow x=2\end{aligned}$$

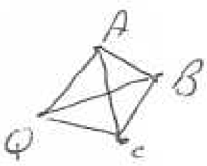
So Q has position vector $(2,3,0)$

A, B and C have position vectors $(7,5,1)$, $(3,5,1)$ and $(3,5,5)$ respectively

let $\underset{\sim}{l}=\overrightarrow{QA}=(7,5,1)-(2,3,0)=(5,2,1)$

$\underset{\sim}{m}=\overrightarrow{QB}=(3,5,1)-(2,3,0)=(1,2,1)$

$\underset{\sim}{n}=\overrightarrow{QC}=(3,5,5)-(2,3,0)=(1,2,5)$



the volume of the tetrahedron QABC is $\frac{1}{6}\left|\underset{\sim}{l}\cdot\underset{\sim}{m}\times\underset{\sim}{n}\right|$

$$=\frac{1}{6}\begin{vmatrix}5&2&1\\1&2&1\\1&2&5\end{vmatrix}=\frac{1}{6}\begin{vmatrix}5&2&1\\1&2&1\\0&0&4\end{vmatrix}=\frac{4}{6}\begin{vmatrix}5&2\\1&2\end{vmatrix}=\frac{4}{6}(10-2)$$

$$=\frac{16}{3}$$