Question 13 solution

a) The homogeneous system of equations

$$2x_1 + 5x_2 = 0 \quad ①$$
$$-x_1 + 3x_2 - 2x_3 = 0 \quad ②$$
$$3x_1 + 13x_2 - 2x_3 = 0 \quad ③$$

can be written in matrix form $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{0}$ where

$$\underset{\sim}{A} = \begin{pmatrix} 2 & 5 & 0 \\ -1 & 3 & -2 \\ 3 & 13 & -2 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

A homogeneous system of equations always has trivial solutions. The system will have non-trivial solutions only if $\det \underset{\sim}{A} = 0$.

Let $\Delta = \det \underset{\sim}{A} = \begin{vmatrix} 2 & 5 & 0 \\ -1 & 3 & -2 \\ 3 & 13 & -2 \end{vmatrix}$

then
$$\left.\begin{aligned} R_1' &= R_1 \\ R_2' &= R_2 - R_3 \\ R_3' &= R_3 \end{aligned}\right\} \rightarrow \quad \Delta = \begin{vmatrix} 2 & 5 & 0 \\ -4 & -10 & 0 \\ 3 & 13 & -2 \end{vmatrix} = (-2)\begin{vmatrix} 2 & 5 \\ -4 & -10 \end{vmatrix} = 0$$

So non-trivial solution exist.

From equation ① $\quad 2x_1 + 5x_2 = 0 \quad$ so $\quad x_2 = -\frac{2}{5}x_1 \quad ④$

Substitute ④ into ② $\rightarrow \quad -x_1 - \frac{6}{5}x_1 - 2x_3 = 0 \rightarrow -\frac{11}{5}x_1 - 2x_3 = 0$

giving $x_3 = -\frac{11}{10}x_1 \quad ⑤$

[Note: x_2 and x_3 defined by ④ and ⑤ automatically satisfy ③]

Hence solutions of the system are

$$x_1 = t, \quad x_2 = -\frac{2}{5}t, \quad x_3 = -\frac{11}{10}t \quad \text{for } -\infty < t < \infty.$$

Question 13 continued

b) $$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\quad + 8x_3 &= 5 \end{aligned}\right\} \rightarrow \underset{\sim}{A}\,\underset{\sim}{X} = \underset{\sim}{Y}$$

where $\underset{\sim}{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$, $\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\underset{\sim}{Y} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

The augmented matrix for the system is $(\underset{\sim}{A} : \underset{\sim}{Y})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 8 & 5 \end{array}\right)$$

Using the row reduction method

$$\left.\begin{aligned} R_1' &= R_1 \\ R_2' &= R_2 - 2R_1 \\ R_3' &= R_3 - R_1 \end{aligned}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & -2 & 5 & 4 \end{array}\right)$$

$$\left.\begin{aligned} R_1' &= R_1 \\ R_2' &= R_2 \\ R_3' &= R_3 + 2R_2 \end{aligned}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

So $x_3 = 0$ and back substitution gives

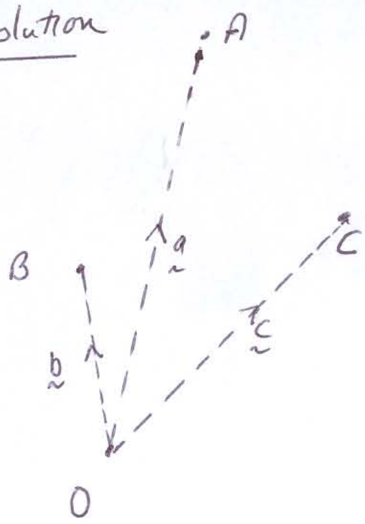
$x_2 = -2$ and $x_1 + 2(-2) + 3(0) = 1 \rightarrow x_1 = 5$

So the solution is

$\underline{x_1 = 5, \quad x_2 = -2, \quad x_3 = 0.}$

Question 14 solution

a)



Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be the position vectors of A, B, and C respectively.
Then $\underset{\sim}{a} = (-2, 1, 3)$, $\underset{\sim}{b} = (20, 5, 0)$ and $\underset{\sim}{c} = (25, 10, 0)$.

The plane containing the points A, B and C is perpendicular to the vector $\underset{\sim}{p}$ where
$\underset{\sim}{p} = \overrightarrow{AB} \times \overrightarrow{BC}$.

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = (20, 5, 0) - (-2, 1, 3) = (22, 4, -3)$

$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b} = (25, 10, 0) - (20, 5, 0) = (5, 5, 0)$

so $\underset{\sim}{p} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 22 & 4 & -3 \\ 5 & 5 & 0 \end{vmatrix} = \underset{\sim}{i}(15) - \underset{\sim}{j}(15) + \underset{\sim}{k}(22 \times 5 - 5 \times 4)$

$= (15, -15, 90)$

The equation of the plane is $\underset{\sim}{r} \cdot \underset{\sim}{p} = \text{const} = K$ that is $(x, y, z) \cdot (15, -15, 90) = K$

$\rightarrow$ $15x - 15y + 90z = K$. The plane passes through A so

$15(-2) - 15(1) + 90(3) = K$ which gives $K = 225$

Hence the equation of the plane containing A, B and C is $15x - 15y + 90z = 225$

ie $\underline{x - y + 6z = 15}$

b) The line which passes through B and C is parallel to $\overrightarrow{BC}$

So the equation is $\underset{\sim}{r} = \underset{\sim}{b} + \lambda \overrightarrow{BC}$, ie $\underline{\underset{\sim}{r} = (20, 5, 0) + \lambda (5, 5, 0)}$

for $-\infty < \lambda < \infty$

c) Given that the position vector of the boat is $(10, 3, 0)$ and its direction is parallel to the vector $(3, 1, 0)$ the equation of the path of the boat is $\underline{\underset{\sim}{r} = (10, 3, 0) + \mu (3, 1, 0)}$ for $-\infty < \mu < \infty$.

Question 14 continued

d) The lines $\underset{\sim}{r} = (20,5,0) + \lambda(5,5,0)$ and $\underset{\sim}{r} = (10,3,0) + \mu(3,1,0)$ cross when

$$(20,5,0) + \lambda(5,5,0) = (10,3,0) + \mu(3,1,0).$$

That is when the components are equal ie.

x comp: $20 + 5\lambda = 10 + 3\mu$ (i)

y comp: $5 + 5\lambda = 3 + \mu$ (ii)

Solving these equations for λ and μ gives

(i) − (ii) $\rightarrow$ $15 = 7 + 2\mu \rightarrow \mu = 4$

Subs into (ii) $\rightarrow$ $\lambda = \dfrac{3+4-5}{5} = \dfrac{2}{5}$

The coordinates of the point of intersection of the lines are therefore

$\underset{\sim}{r} = (20,5,0) + \frac{2}{5}(5,5,0) = \underline{(22,7,0)}$

[Note: the same result is given by $\underset{\sim}{r} = (10,3,0) + 4(3,1,0) = (22,7,0)$]

Question 15 solution

The eigenvalues of a matrix $\underset{\sim}{A}$ are the solutions of the equation $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$ (called the characteristic equation).

Given that $\underset{\sim}{A} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

We need to solve the equation

$$\Delta = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\left.\begin{array}{l} R_1' = R_1 + R_3 \\ R_2' = R_2 \\ R_3' = R_3 \end{array}\right\} \rightarrow \quad \Delta = \begin{vmatrix} 3-\lambda & 0 & 3-\lambda \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & 0 & 1 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$

$$\left.\begin{array}{l} C_1' = C_1 - C_3 \\ C_2' = C_2 \\ C_3' = C_3 \end{array}\right\} \rightarrow \quad \Delta = (3-\lambda) \begin{vmatrix} 0 & 0 & 1 \\ 4 & 2-\lambda & -1 \\ 3+\lambda & 1 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 4 & 2-\lambda \\ 3+\lambda & 1 \end{vmatrix} = (3-\lambda)\left[4 - (3+\lambda)(2-\lambda)\right]$$

$$= (3-\lambda)(\lambda^2 + \lambda - 2)$$

$$= (3-\lambda)(\lambda+2)(\lambda-1)$$

So $\Delta = 0$ has solutions $\underline{\lambda = 1, -2, 3}$. These are the eigenvalues of Matrix $\underset{\sim}{A}$

Question 15 continued

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is obtained from solving the system of equations $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{X}$

ie
$$\left.\begin{aligned} x_1 - x_2 + 4x_3 &= x_1 \\ 3x_1 + 2x_2 - x_3 &= x_2 \\ 2x_1 + x_2 - x_3 &= x_3 \end{aligned}\right\} \rightarrow \begin{aligned} -x_2 + 4x_3 &= 0 \quad (1) \\ 3x_1 + x_2 - x_3 &= 0 \quad (2) \\ 2x_1 + x_2 - 2x_3 &= 0 \quad (3) \end{aligned}$$

from (1) we see that $x_2 = 4x_3$ (4)

from (2) & (4) $3x_1 + 3x_3 = 0 \rightarrow x_1 = -x_3$ (5)

equation (3) is automatically satisfied if (4) & (5) hold.

If $x_3 = t$ then $x_1 = -t$ and $x_2 = 4t$ for all t

hence the eigenvector corresponding to eigenvalue $\lambda = 1$ is

$\underset{\sim}{v}_1 = (-1, 4, 1)^T$.

The eigenvector corresponding to eigenvalue $\lambda = 2$ is obtained from solving the system of equations

$$3x_1 - x_2 + 4x_3 = 0 \quad (1)$$
$$3x_1 + 4x_2 - x_3 = 0 \quad (2)$$
$$2x_1 + x_2 + x_3 = 0 \quad (3)$$

(1) − (2) → $-5x_2 + 5x_3 = 0 \rightarrow x_2 = x_3$ (4)

subs into (1) → $3x_1 + 3x_3 = 0 \rightarrow x_1 = -x_3$ (5)

(3) is automatically satisfied if (4) & (5) hold. Hence the eigenvector corresponding to eigenvalue $\lambda = -2$ is

$\underset{\sim}{v}_2 = (-1, 1, 1)^T$.

Question 16 solution

a) The system of equations

$$2x_1 - 3x_2 + 3x_3 = 3$$
$$4x_1 + 6x_2 + 9x_3 = -1$$
$$2x_1 + x_2 + 4x_3 = 1$$

Can be written in matrix form $\underset{\sim}{A}\underset{\sim}{X} = \underset{\sim}{Y}$ where

$$\underset{\sim}{A} = \begin{pmatrix} 2 & -3 & 3 \\ 4 & +6 & 9 \\ 2 & 1 & 4 \end{pmatrix}, \quad \underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \underset{\sim}{Y} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

The system will only have a unique solution if $\det \underset{\sim}{A} \neq 0$

Let $\Delta = \det \underset{\sim}{A} = \begin{vmatrix} 2 & -3 & 3 \\ 4 & 6 & 9 \\ 2 & 1 & 4 \end{vmatrix}$

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}\right\} \to \Delta = \begin{vmatrix} 2 & -3 & 3 \\ 0 & 12 & 3 \\ 0 & 4 & 1 \end{vmatrix}$$

$$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_3 - \frac{1}{3}R_2 \end{array}\right\} \to \Delta = \begin{vmatrix} 2 & -3 & 3 \\ 0 & 12 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Hence the system does NOT have a unique solution.

Now form the augmented matrix $(\underset{\sim}{A} : \underset{\sim}{Y})$

$$\left(\begin{array}{ccc|c} 2 & -3 & 3 & 3 \\ 4 & 6 & 9 & -1 \\ 2 & 1 & 4 & 1 \end{array}\right)$$

Carrying out the row operations carried out above we obtain

$$\left(\begin{array}{ccc|c} 2 & -3 & 3 & 3 \\ 0 & 12 & 3 & -7 \\ 0 & 4 & 1 & -2 \end{array}\right) \text{ followed by } \left(\begin{array}{ccc|c} 2 & -3 & 3 & 3 \\ 0 & 12 & 3 & -7 \\ 0 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

Question 1b continued

from which we see that $0x_3 = 1/3$. There is no solution to this equation, hence the system does <u>NOT</u> have a solution ie the equations are inconsistent.

b) Given that $\underset{\sim}{r} = (e^t + e^{-t})\underset{\sim}{i} + (e^t - e^{-t})\underset{\sim}{j} + t^2\underset{\sim}{k}$ for $t \geq 0$

then $\underset{\sim}{v} = \dot{\underset{\sim}{r}} = (e^t - e^{-t})\underset{\sim}{i} + (e^t + e^{-t})\underset{\sim}{j} + 2t\underset{\sim}{k}$

and $\underset{\sim}{a} = \ddot{\underset{\sim}{r}} = (e^t + e^{-t})\underset{\sim}{i} + (e^t - e^{-t})\underset{\sim}{j} + 2\underset{\sim}{k}$

(i) $\underset{\sim}{r} \cdot \underset{\sim}{v} = (e^{2t} - e^{-2t}) + (e^{2t} - e^{-2t}) + 2t^3 = 2(e^{2t} - e^{-2t}) + 2t^3$

$= 0$ when $t = 0$

So $\underset{\sim}{r}$ and $\underset{\sim}{v}$ are orthogonal vectors at $t = 0$.

(ii) $\underset{\sim}{v} \cdot \underset{\sim}{a} = (e^{2t} - e^{-2t}) + (e^{2t} - e^{-2t}) + 4t = 2(e^{2t} - e^{-2t}) + 4t$

$= 0$ when $t = 0$

(iii) $\underset{\sim}{r} - \underset{\sim}{a} = (t^2 - 2)\underset{\sim}{k}$ and $\underset{\sim}{v} \cdot (\underset{\sim}{r} - \underset{\sim}{a}) = 2t(t^2 - 2) = 0$ when $t = 0$ and $t = \pm\sqrt{2}$. However $t \geq 0$ so $\underset{\sim}{r}$ and $\underset{\sim}{r} - \underset{\sim}{a}$ are orthogonal for $t = 0$ and $t = \sqrt{2}$.