Section A

1. The value of the determinant

$$\begin{vmatrix} 2 & -1 & 3 \\ -4 & 2 & 3 \\ -2 & 5 & 0 \end{vmatrix}$$

is

(a) -96, (b) -84, (c) -72, (d) 108.

2. The force $\underline{F}$ which has magnitude 6 and direction parallel to the vector $(3, 1, -1)$ has components

(a) $\left(\frac{18}{\sqrt{11}}, \frac{6}{\sqrt{11}}, \frac{-6}{\sqrt{11}}\right)$, (b) $\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right)$,

(c) $(3\sqrt{11}, \sqrt{11}, -\sqrt{11})$, (d) $(18, 6, -6)$.

3. The angle between the vectors $\underline{u} = 2\underline{i} - 3\underline{j} + 6\underline{k}$ and $\underline{v} = 2\underline{i} - 2\underline{j} - 3\underline{k}$ is

(a) $\cos^{-1}\left(\frac{-20}{7\sqrt{17}}\right)$, (b) $\cos^{-1}\left(\frac{-8}{7\sqrt{17}}\right)$, (c) $\cos^{-1}\left(\frac{-20}{\sqrt{119}}\right)$, (d) $\cos^{-1}\left(\frac{-8}{119}\right)$.

4. If the position vectors of the points A and B are $(4, -1, 3)$ and $(10, 8, -2)$ respectively, then the position vector of the point P on the line joining A to B such that $AP = \frac{1}{3} AB$ is

(a) $(12, 11, -11/3)$, (b) $(2, -4, 14/3)$, (c) $(8, 5, -1/3)$, (d) $(6, 2, 4/3)$.

5. If $\underline{u} = (2,3,1)$ and $\underline{v} = (-1,4,-3)$ then $\underline{u} \times \underline{v}$ has components
(a) $(-5,5,11)$, (b) $(13,-5,-11)$, (c) $(-13,5,11)$, (d) $(-13,-5,-11)$.

6. For any two vectors $\underline{p}$ and $\underline{q}$, the expression $(6\underline{p} + 3\underline{q}) \times (\underline{q} + 2\underline{p})$ can be simplified to

(a) $6\,\underline{p} \times \underline{q}$, (b) $12\,\underline{p} \times \underline{q}$, (c) $-12\,\underline{p} \times \underline{q}$, (d) $\underline{0}$.

7. The cartesian equation of the plane which is parallel to the vectors $\underline{i} + \underline{k}$ and $\underline{i} + \underline{j}$ and passes through the point $(3,1,1)$ is
(a) $2x + y + z = 8$, (b) $x - y - z = 1$,
(c) $3x + y + z = 1$, (d) $x + y + z = 5$.

8. If the matrices A, B and C are defined by
$$A = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 \\ 1 & 2 \\ -1 & -1 \end{pmatrix} \text{ and } C = AB$$

then the element C_{22} is
(a) 0, (b) 4, (c) 6, (d) 7.

9. If the matrices A, B and C are defined by
$$A = \begin{pmatrix} 2 & x \\ 2x & 1 \end{pmatrix} \quad B = \begin{pmatrix} y & 2 \\ 1 & -1 \end{pmatrix} \text{ and } C = A + B,$$

where x and y are constants, then C is an anti-symmetric matrix only when
(a) $x = 1$ and for all values of y, (b) $x = 3$, $y = -2$,
(c) $x = -1$, $y = -2$, (d) $x = -1$ and for all values of y.

10. If the matrix A and its inverse A^{-1} are defined to be

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & x & \frac{1}{2} \\ 1 & 0 & 0 \\ y & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

where x and y are constants, then

(a) $x = -\frac{1}{2}$, $y = -\frac{1}{2}$, (b) $x = -\frac{1}{2}$, $y = \frac{1}{2}$,

<u>(c)</u> $x = \frac{1}{2}$, $y = -\frac{1}{2}$, (d) $x = \frac{1}{2}$, $y = \frac{1}{2}$.

11. Given that $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ are eigenvectors of the matrix $\begin{pmatrix} 2 & -3 \\ -10 & 1 \end{pmatrix}$,

the corresponding eigenvalues are

(a) $\frac{1}{4}$ and $\frac{1}{7}$, (b) $-\frac{1}{4}$ and $\frac{1}{7}$, (c) 4 and 7, <u>(d)</u> -4 and 7.

12. If the Jacobi iterative method is applied to the system of equations

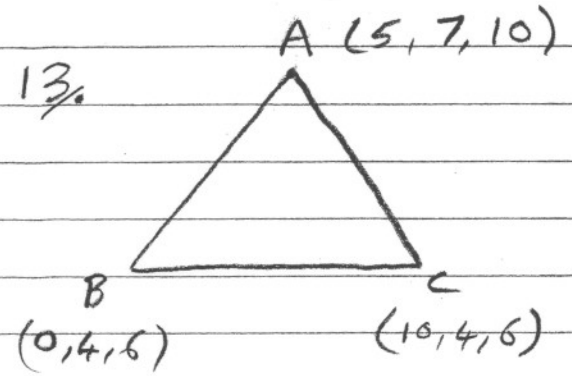
$$\begin{aligned} 2x + y + z &= 3 \\ x + 2y + z &= 4 \\ x + y + 2z &= 1, \end{aligned}$$

Starting from the initial guess $x = y = z = 1$, the result after one step of the iteration is

(a) $x = 3/2$, $y = 2$, $z = 1/2$, (b) $x = 1$, $y = 2$, $z = -1$,

<u>(c)</u> $x = 1/2$, $y = 1$, $z = -1/2$, (d) $x = 3$, $y = 4$, $z = 1$.

13.



$\vec{AB} = (0,4,6) - (5,7,10)$
$= (-5,-3,-4)$
$\vec{BC} = (10,4,6) - (0,4,6)$
$= (10,0,0)$

(a) Area of triangle $= \frac{1}{2} |\vec{AB} \times \vec{BC}|$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & -3 & -4 \\ 10 & 0 & 0 \end{vmatrix} = 0\underline{i} - 40\underline{j} + 30\underline{k}$$

$|\vec{AB} \times \vec{BC}| = \sqrt{40^2 + 30^2} = \sqrt{1600 + 900} = \sqrt{2500} = 50$

5 So area $= 25$.

(b) Normal vector $= \vec{AB} \times \vec{BC} = -40\underline{j} + 30\underline{k}$

3 So unit normal $\hat{\underline{n}} = (0, -4/5, 3/5)$. (or minus this)

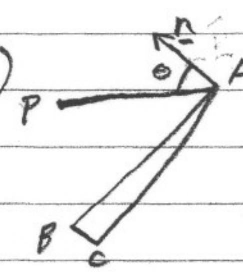
(c) Equation of plane is $\underline{r} \cdot \hat{\underline{n}} = c$
where $\underline{r} = (x, y, z)$. So $-\frac{4}{5}y + \frac{3}{5}z = c$.
Plane must pass through $B = (0, 4, 6)$
So $-16/5 + 18/5 = c = 2/5$.

4 Equation is $-4y + 3z = 2$.

(d) $\vec{AP} = (12, 11, 14) - (5, 7, 10) = (7, 4, 4)$.
Length of wire $= |\vec{AP}| = \sqrt{7^2 + 4^2 + 4^2} = \sqrt{49 + 16 + 16} = \sqrt{81} = 9$.

3

(e)



$\vec{AP} \cdot \hat{\underline{n}} = |\vec{AP}|\,|\hat{\underline{n}}| \cos\theta$

$(7,4,4) \cdot (0, -4/5, 3/5) = 9 \times 1 \times \cos\theta$

$\Rightarrow \cos\theta = (-16/5 + 12/5)/9 = -4/45$

$\theta = \cos^{-1}(-4/45) = 95.1° = 1.66$ rad.

θ is angle between wire and normal $\hat{\underline{n}}$.

5 Angle between wire and roof $= 90 - \theta = \pm 5.1° = \pm .089$ rad.
(either sign acceptable).

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14. (a) $\underline{r} = (2t + \cos t)\,\underline{i} + (t + \sin t)\,\underline{j} + (2 - \sin t)\,\underline{k}.$

2 (i) $\underline{v} = \dfrac{d\underline{r}}{dt} = (2 - \sin t)\,\underline{i} + (1 + \cos t)\,\underline{j} - \cos t\,\underline{k}.$

2 $\underline{a} = \dfrac{d\underline{v}}{dt} = -\cos t\,\underline{i} - \sin t\,\underline{j} + \sin t\,\underline{k}$

(ii) At $t = 0$, $\underline{v} = 2\underline{i} + 2\underline{j} - \underline{k}.$

3 Speed $= |\underline{v}| = \sqrt{2^2 + 2^2 + 1^2} = 3.$

(iii) At $t = 0$, $\underline{r} = 1\underline{i} + 0\underline{j} + 2\underline{k}.$

So $\underline{r} \cdot \underline{v} = 1 \times 2 + 0 \times 2 + 2 \times -1 = 0$,

3 hence $\underline{r}$ and $\underline{v}$ are perpendicular.

(iv) force = mass × acceleration, so force is greatest when acceleration is greatest.

$|\underline{a}| = \sqrt{(\cos^2 t + \sin^2 t + \sin^2 t)} = \sqrt{(1 + \sin^2 t)}.$

3 This is greatest when $\sin t = \pm 1$, so $t = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(b) $f(x, y, z) = x^2 y^2 - y^3 + xz.$

2 3 $\underline{u} = \underline{\nabla} f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right) = (2xy^2 + z, 2x^2 y - 3y^2, x)$

2 and $\underline{\nabla} \cdot \underline{u} = \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} + \dfrac{\partial u_3}{\partial z} = 2y^2 + 2x^2 - 6y.$

So at $x = 1$, $y = 1$, $z = 2$,

2 1 $\underline{u} = (2 + 2, 2 - 3, 1) = (4, -1, 1)$

1 and $\underline{\nabla} \cdot \underline{u} = 2 + 2 - 6 = -2$

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15. (a) $x - y - z = 0$ ①

$3x + y = 0$ ②

$6x - 2y - 3z = 0$ ③

Determinant is $\begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 0 \\ 6 & -2 & -3 \end{vmatrix} = 1\times(-3) + 1\times(-9) - 1\times(-6-6)$
$= -3 - 9 + 12 = 0$

4

So there are infinitely many solutions.
We can choose $x = t$, then $y = -3t$ (from ②)

3 and $z = x - y$ (from ①) $= 4t$.

(b) $\begin{pmatrix} 1 & 5 & -3 \\ 2 & 9 & -14 \\ 1 & -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 23 \\ -5 \end{pmatrix}$ in matrix form.

2

Augmented matrix is $\left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 2 & 9 & -14 & 23 \\ 1 & -5 & 5 & -5 \end{array}\right)$

1

Row operations:

$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}\right\} \Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 0 & -1 & -8 & -11 \\ 0 & -10 & 8 & -22 \end{array}\right)$

3

$\left.\begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_3 - 10R_2 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 0 & -1 & -8 & -11 \\ 0 & 0 & 88 & 88 \end{array}\right)$

3

$88z = 88 \Rightarrow z = 1$

$-y - 8 = -11 \Rightarrow y = 3$

$x + 5y - 3z = 17 \Rightarrow x + 15 - 3 = 17 \Rightarrow x = 5$

4 So Solution is $x = 5$, $y = 3$, $z = 1$.

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16. To find eigenvalues, solve $|M - \lambda I| = 0$

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$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 4 \\ 0 & 4 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 0 & 2-\lambda \end{vmatrix}$$

$= (2-\lambda)(4-4\lambda+\lambda^2-16) - 3(6-3\lambda)$

$= -\lambda^3+4\lambda^2+2\lambda^2-8\lambda+12\lambda-24 \quad -18+9\lambda$

$= -\lambda^3+6\lambda^2+13\lambda-42$

$= (2-\lambda)(\lambda^2-4\lambda-21)$

6 $= (2-\lambda)(\lambda+3)(\lambda-7) = 0$

So eigenvalues are $\lambda = 2, \lambda = -3, \lambda = 7.$

[Quicker: $x = 2-\lambda$
$x(x^2-16) - 9x = x(x^2-25) = 0$
but students unlikely to spot this.]

For $\lambda = 2$, to find the eigenvector we need to solve $(M - \lambda I)\underline{v} = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3y = 0, \quad 3x + 4z = 0, \quad 4y = 0$

5 So $y = 0$ and $z = -\frac{3}{4}x \Rightarrow \underline{v} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ or any multiple

For $\lambda = 7$, $\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-5x + 3y = 0, \quad 3x - 5y + 4z = 0, \quad 4y - 5z = 0.$

We can choose $x = 1$. Then $y = \frac{5}{3}, \; z = \frac{4}{3}$

5 $\underline{v} = \begin{pmatrix} 1 \\ 5/3 \\ 4/3 \end{pmatrix}$ or $\underline{v} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ or any multiple.

The dot product of the two eigenvectors is

2 $4 \times 3 + 0 \times 5 - 3 \times 4 = 0$ So they are orthogonal.

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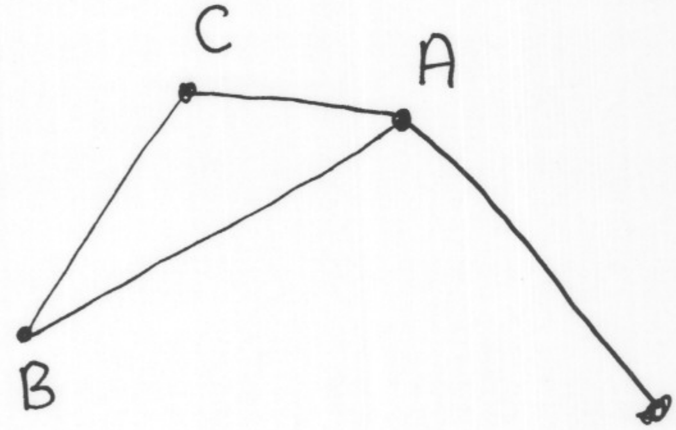
HGIM02 2005-6 SOLUTIONS

Q13



(a) $$\text{area} = \frac{1}{2}\left|\vec{AB}\times\vec{BC}\right| \quad \left(\text{or } \frac{1}{2}\left|\vec{AC}\times\vec{BC}\right| \text{ or } \frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|\right)$$

where

$$\vec{AB} = (0,4,6)-(5,7,10) = (-5,-3,-4)$$

$$\vec{BC} = (10,4,6)-(0,4,6) = (10,0,0)$$

$$\vec{AB}\times\vec{BC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & -3 & -4 \\ 10 & 0 & 0 \end{vmatrix} = 0\underline{i} - 40\underline{j} + 30\underline{k} = (0,-40,30)$$

$$\text{area} = \frac{1}{2}\left|-40\underline{j}+30\underline{k}\right| = \frac{1}{2}\sqrt{40^2+30^2} = \frac{1}{2}\sqrt{1600+900} = \frac{1}{2}50 = 25$$

(b) $\vec{AB}\times\vec{BC}$ $\left(\text{or } \vec{AC}\times\vec{BC} \text{ or } \vec{AB}\times\vec{AC}\right)$ is a vector normal to the roof. Corresponding unit vector is

$$\hat{n} = \frac{\vec{AB}\times\vec{BC}}{\left|\vec{AB}\times\vec{BC}\right|} = \frac{-40\underline{j}+30\underline{k}}{50} = -\frac{4}{5}\underline{j}+\frac{3}{5}\underline{k}$$

13 (c) Equation of the plane is

$$\underline{r}\cdot\hat{n} = \left(0,-\frac{4}{5},\frac{3}{5}\right)\cdot(x,y,z)$$

$$= -\frac{4}{5}y + \frac{3}{5}z \quad = \text{const}$$

Or, $-4y+3z=c$ where c is determined by substituting explicit values for (x,y,z) such as

$(x,y,z)=(5,7,10) \Rightarrow -4\cdot 7 + 3\cdot 10 = 2 = c$

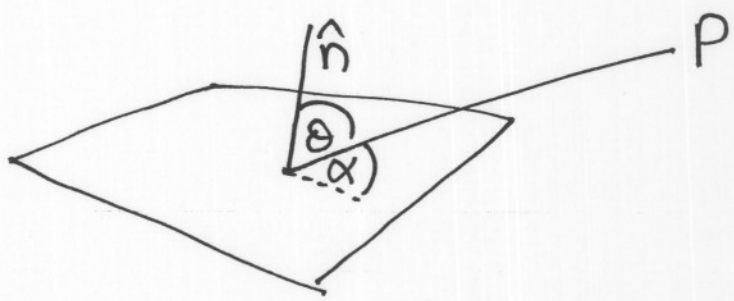
so equation of the plane is $-4y+3z=2$.

(d) $\vec{AP} = (12,11,14)-(5,7,10) = (7,4,4)$

distance $= |\vec{AP}| = \sqrt{7^2+4^2+4^2} = \sqrt{49+16+16}$

$= \sqrt{81} = 9.$

(e)



$\hat{n}\cdot\vec{AP} = |\hat{n}|\,|\vec{AP}|\cos\theta$ $\quad\theta$ = angle between <u>normal</u> and wire

$\Rightarrow -\frac{4}{5}\cdot 4 + \frac{3}{5}\cdot 4 = 1\cdot 9\cdot\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{9}\left(-\frac{1}{5}\right) = -\frac{1}{45} \Rightarrow \theta = 95.1^\circ$

Angle between <u>plane</u> and wire is

$\alpha = 90^\circ - \theta = -5.1^\circ$ (or 5.1° is acceptable).

13 Common mistakes

(a) • Using $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ in crossproducts instead of $\vec{AB}$, $\vec{BC}$ and $\vec{AC}$.
• Using the dot product instead of the cross product.

(b) Many students, after evaluating $\vec{AB} \times \vec{BC}$ etc correctly in part (a), start again from scratch in part (b). Although this gets full marks (if done right) it is an inefficient use of time.

(d) Using $\vec{OP}$ instead of $\vec{AP}$.

(e) Again, using $\vec{OP}$ instead of $\vec{AP}$ and using other vectors such as $\vec{OA}$, $\vec{AB}$ instead of $\hat{n}$.

Q14 (a) (i) $\underline{v} = \frac{d\underline{r}}{dt}$

$= (2-\sin t)\underline{i} + (1+\cos t)\underline{j} - \cos t\,\underline{k}$

$\underline{a} = \frac{d\underline{v}}{dt}$

$= -\cos t\,\underline{i} - \sin t\,\underline{j} + \sin t\,\underline{k}$

(ii) Initial velocity is

$\underline{v}(0) = (2-0)\underline{i} + (1+1)\underline{j} - (1)\underline{k}$

$= 2\underline{i} + 2\underline{j} - \underline{k} \quad (= (2,2,-1))$

Initial speed is

$s(0) = |\underline{v}(0)| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$

(iii) $\underline{r}(0) = (0+1)\underline{i} + (0+0)\underline{j} + (2-0)\underline{k}$

$= \underline{i} + 2\underline{k} \quad (= (1,0,2))$

$\underline{r}(0) \cdot \underline{v}(0) = 2 + 0 - 2 = 0$

$\Rightarrow$ $\underline{r}(0)$ and $\underline{v}(0)$ are perpendicular

(iv) By Newton's 3rd law $F = ma$, maximum force coincides with maximum accelaration:

$|\underline{a}| = \sqrt{\cos^2 t + \sin^2 t + \sin^2 t} = \sqrt{1+\sin^2 t}$

14 This is a maximum when $\sin^2 t = 1$ or $\sin t = \pm 1$

$$\Rightarrow t = \frac{\pi}{2} + n\pi \qquad n = 0, 1, 2, \cdots$$

(b) $$\underline{u} = \nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

$$= (2xy^2 + z)\underline{i} + (2x^2y - 3y^2)\underline{j} + x\underline{k}$$

When $(x, y, z) = (1, 1, 2)$ $\underline{u} = (2+2)\underline{i} + (2-3)\underline{j} + \underline{k}$

$$= 4\underline{i} - \underline{j} + \underline{k}$$

$$= (4, -1, 1)$$

$$\nabla \cdot \underline{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$= (2y^2) + (2x^2 - 6y) + (0)$$

$$= 2y^2 + 2x^2 - 6y$$

When $(x, y, z) = (1, 1, 2)$, $\nabla \cdot \underline{u} = 2 + 2 - 6 = -2$

14 Common mistakes

(a) (ii) Finding the accelaration $\underline{a} = \frac{dv}{dt}$ instead of the speed $s = |\underline{v}|$ at $t=0$. Motto: read the question.

(iv) Maximising speed etc and not accelaration. Working with the vector accelaration and not its magnitude.

(b) Writing $\underline{u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ (a scalar, not a vector). Confusing $\nabla \cdot \underline{u}$ with $\nabla \times \underline{u}$.

15. (a)

$$x - y - z = 0 \quad (1)$$
$$3x + y = 0 \quad (2)$$
$$6x - 2y - 3z = 0 \quad (3)$$

The determinant of the system is

$$\begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 0 \\ 6 & -2 & -3 \end{vmatrix} = 1\times(-3) + 1\times(-9) - 1\times(-6-6)$$
$$= -3 - 9 + 12 = 0$$

So there are infinitely many solutions.
We can choose $x = t$. Then (2) $\Rightarrow y = -3t$
and (1) $\Rightarrow z = x - y = 4t$

(b) In matrix form, $\begin{pmatrix} 1 & 5 & -3 \\ 2 & 9 & -14 \\ 1 & -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 23 \\ -5 \end{pmatrix}$.

The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 2 & 9 & -14 & 23 \\ 1 & -5 & 5 & -5 \end{array}\right)$.

Row operations to make rows 2 and 3 start with 0:

$$\left.\begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}\right\} \Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 0 & -1 & -8 & -11 \\ 0 & -10 & 8 & -22 \end{array}\right).$$

Row operation to make row 3 start with 0 0:

$$R_3' = R_3 - 10R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 17 \\ 0 & -1 & -8 & -11 \\ 0 & 0 & 88 & 88 \end{array}\right).$$

$88z = 88 \Rightarrow z = 1$

$-y - 8z = -11 \Rightarrow y = 11 - 8z = 3$

$x + 5y - 3z = 17 \Rightarrow x = 17 - 5y + 3z = 5$

So $x = 5$, $y = 3$, $z = 1$.

16. To find eigenvalues, solve $|M-\lambda I| = 0$ for λ.

$$|M-\lambda I| = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 4 \\ 0 & 4 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)((2-\lambda)^2-16) - 3(3)(2-\lambda).$$

Notice that $2-\lambda$ is a factor. Do <u>not</u> multiply this out, keep it as a factor.

$$|M-\lambda I| = (2-\lambda)(4-4\lambda+\lambda^2-16-9)$$
$$= (2-\lambda)(\lambda^2-4\lambda-21)$$
$$= (2-\lambda)(\lambda-7)(\lambda+3) = 0$$

So the eigenvalues are
<u>$\lambda = 2$, $\lambda = 7$ and $\lambda = -3$.</u>

Now to find the eigenvectors we must find a vector $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $(M-\lambda I)\underline{v} = 0$

For $\lambda = 2$, $\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow 3y = 0$, $3x + 4z = 0$, $4y = 0$.
So y must be zero but we can choose any x and z as long as $3x + 4z = 0$.

16 For example choose $x = 1$, $z = -\frac{3}{4}$

$$V = \begin{pmatrix} 1 \\ 0 \\ -3/4 \end{pmatrix} \quad \text{or} \quad V = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \quad \text{or any multiple.}$$

For $\lambda = 7$, $\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-5x + 3y = 0$, (1)

$3x - 5y + 4z = 0$, (2)

$4y - 5z = 0$, (3)

We can choose $x = 1$.

Then $y = 5/3$ and $z = \frac{4}{3}$ using (1), (3).

check (2): $3x - 5y + 4z = 3 - \frac{25}{3} + \frac{16}{3} = 3 - \frac{9}{3} = 0$.

So $V = \begin{pmatrix} 1 \\ 5/3 \\ 4/3 \end{pmatrix}$ or $V = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ or any multiple.

To show the eigenvectors are orthogonal (perpendicular) we take their dot product.

$$\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = 4 \times 3 + 0 \times 5 - 3 \times 4 = 0$$

So the vectors are orthogonal.