HG1M02—Applied Algebra for Engineers

Coursework 1—Solutions

1. The given determinants are:

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2\times2-(-3)\times1 = 4+3 = 7;$$

$$\begin{vmatrix} 1 & -3 \\ -4 & 2 \end{vmatrix} = 1\times2-(-3)\times(-4) = 2-12 = -10;$$

and

$$\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} = 2\times(-4)-1\times1 = -8-1 = -9.$$

From the results in your notes, these are the determinants that occur in the formal solution of

$$\begin{aligned} 2x-3y &= 1, \\ x+2y &= -4; \end{aligned}$$

and so we can write down $x = -10/7$, $y = -9/7$. No real excuse for getting any of these wrong, as you can check your answers by substituting in the given equations, or by solving the equations by the usual elimination method. 2 marks for each determinant, and 2 each for x and y, total 10; a gift. No marks for solving for x and y any other way!

2. (a) Expanding by the second column,

$$\begin{vmatrix} \sqrt{2} & 0 & -3 \\ 2 & 0 & 0 \\ \pi^2 & 1 & \mathrm{e}^{-3} \end{vmatrix} = -0+0-1\times\begin{vmatrix} \sqrt{2} & -3 \\ 2 & 0 \end{vmatrix} = -(0-(-3)\times2) = -6.$$

We could equally have expanded by the second row, or in other ways.

(b) Lots of ways to do this. For example, we could subtract the third column from the first column and then from the second column to give

$$\begin{vmatrix} 110 & 117 & 110 \\ 111 & 118 & 118 \\ 112 & 119 & 105 \end{vmatrix} = \begin{vmatrix} 0 & 117 & 110 \\ -7 & 118 & 118 \\ 7 & 119 & 105 \end{vmatrix} = \begin{vmatrix} 0 & 7 & 110 \\ -7 & 0 & 118 \\ 7 & 14 & 105 \end{vmatrix} = 7\times7\times\begin{vmatrix} 0 & 1 & 110 \\ -1 & 0 & 118 \\ 1 & 2 & 105 \end{vmatrix}.$$

Now, for example, add the third row to the second and expand by the first column:

$$49\times\begin{vmatrix} 0 & 1 & 110 \\ -1 & 0 & 118 \\ 1 & 2 & 105 \end{vmatrix} = 49\times\begin{vmatrix} 0 & 1 & 110 \\ 0 & 2 & 223 \\ 1 & 2 & 105 \end{vmatrix} = 49\times\begin{vmatrix} 1 & 110 \\ 2 & 223 \end{vmatrix} = 49\times\begin{vmatrix} 1 & 110 \\ 0 & 3 \end{vmatrix} = 49\times3 = 147.$$

where we subtracted twice row one from row two near the end. Or we could, earlier, have subtracted 110× column 2 from column 3 and expanded by the first row. Or at the start we could have done similar things with rows. All roads should lead to the same value!

5 marks for each determinant. No marks for wrong answers, but any evaluation route OK—it's your own problem if you did too much work!

3. We can simply evaluate the determinant directly, but it is worth getting some zeros in it first. For example, adding twice row 3 to row 2, we have

$$\begin{vmatrix} \lambda-1 & 2 & 3 \\ 2 & \lambda+2 & 6 \\ -1 & 1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 2 & 3 \\ 0 & \lambda+4 & 2\lambda+8 \\ -1 & 1 & \lambda+1 \end{vmatrix} = (\lambda+4)\times \begin{vmatrix} \lambda-1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & \lambda+1 \end{vmatrix}.$$

Now, for example, subtract twice column 2 from column 3, and then expand by the second row; so the original determinant is

$$(\lambda+4)\begin{vmatrix} \lambda-1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & \lambda-1 \end{vmatrix} = (\lambda+4)\times 1 \times \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda+4)\,((\lambda-1)^2-1) = (\lambda+4)\,\lambda\,(\lambda-2)$$

[7 marks]. You didn't need to evaluate it exactly this way, of course; again, there are many possible routes.

So the determinant is zero when $\lambda = -4$ or 0 or 2 [1 mark for each, total 10]. This may seem like a somewhat pointless problem, but it is very similar to an important process that we'll see later.

4. As E is the mid-point of AB, we have $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB}$, and so $\boldsymbol{e} = \boldsymbol{a}+\overrightarrow{AE} = \boldsymbol{a}+\frac{1}{2}\overrightarrow{AB} = \boldsymbol{a}+\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$, as required [3 marks]. Similarly, $\boldsymbol{f} = \frac{1}{2}(\boldsymbol{c}+\boldsymbol{d})$ [2 marks], and $\boldsymbol{g} = \frac{1}{2}(\boldsymbol{e}+\boldsymbol{f}) = \frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$ [2 marks].

Because of the symmetry, we will get to the same point if we take the mid-point of the line joining the mid-points of AC and BD or of AD and BC. So these three lines are concurrent [3 marks]. [In fact, G is the centre of gravity of equal weights at the four vertices, and we are just finding this same point in three different ways.]

10 marks available for each question, total 40.