HG1M02—Applied Algebra for Engineers

Coursework 2—Solutions

1. *(a)* The given vectors $2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ have lengths $\sqrt{2^2+1^2+(-1)^2} = \sqrt{6}$ and [similarly] $\sqrt{6}$, and scalar product $2\times1+1\times(-1)+(-1)\times2 = -1$, so the angle, θ, between them satisfies $\sqrt{6}\times\sqrt{6}\times\cos\theta = -1$, so $\theta = \cos^{-1} -\frac{1}{6} \approx 1.74$ radians, 100 degrees. [**3 marks**]

   (b) If we orient the axes as suggested, then we can take the common vertex to be the origin, the opposite corner to be $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, a typical edge to run from the origin to $\boldsymbol{i}$, and a typical face diagonal to run from the origin to $\boldsymbol{i}+\boldsymbol{j}$ [or $\boldsymbol{i}+\boldsymbol{k}$ or $\boldsymbol{j}+\boldsymbol{k}$. So the long diagonal has length $\sqrt{3}$, face diagonals have length $\sqrt{2}$, and edges length 1; and all relevant dot products are 1 or 2. So, by the same method as above, the angle between: (i) a long diagonal and an edge is $\cos^{-1}(1/\sqrt{3}) \approx 0.96$ radians, 55 degrees [**2 marks**]; (ii) the long diagonal and a face diagonal is $\cos^{-1}(2/(\sqrt{3}\sqrt{2})) \approx 0.61$ radians, 35 degrees [**3 marks**]; and (iii) two face diagonals is $\cos^{-1}(1/(\sqrt{2}\sqrt{2})) = \pi/3$ radians, 60 degrees ['obvious' with hindsight, as the face diagonals join up into equilateral triangles] [**2 marks**; total 10].

2. The vector from $(0,1,2) = \boldsymbol{j}+2\boldsymbol{k}$ to $(1,-1,-1) = \boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$ is $\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$. So *(a)* a vector form of the equation of the line is $\boldsymbol{r} = \boldsymbol{j}+2\boldsymbol{k}+\lambda(\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k})$ [or $\boldsymbol{r} = (0,1,2)+\lambda(1,-2,-3)$ or other equivalent forms] [**3 marks**] and *(b)* the cartesian form is given by setting $\boldsymbol{r} = (x,y,z)$ and 'solving' each component for λ:

$$x = \frac{y-1}{-2} = \frac{z-2}{-3} \; [\, = \lambda]$$

   [**3 marks**].

   To find the closest approach to the x-axis, we first need a common normal, $(\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k})\times\boldsymbol{i} = 2\boldsymbol{k}-3\boldsymbol{j}$. A typical point on the x-axis is $\boldsymbol{0}$ and on the line is $\boldsymbol{j}+2\boldsymbol{k}$, so the closest approach is the component of $\boldsymbol{j}+2\boldsymbol{k}$ in the direction of $-3\boldsymbol{j}+2\boldsymbol{k}$, which is their dot product, $-3+4=1$, divided by the size of $-3\boldsymbol{j}+2\boldsymbol{k}$, or $\sqrt{(-3)^2+2^2} = \sqrt{13}$. So the closest approach is $1/\sqrt{13}$ [$=\sqrt{13}/13 \approx 0.28$] [**4 marks**; total 10].

   You could equally get -0.28, if you happened to choose the opposite order for the directions, or for the points on the line, but of course you should then discard the sign to get the absolute distance. Perfectly OK, but harder work, to use calculus on the distance between an arbitrary point on the x-axis and an arbitrary point on the line.

3. Similarly to the previous question, we can find two vectors in the plane by subtracting [any] one point from the other two; for example, $2\boldsymbol{i}+3\boldsymbol{k}-(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = \boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $3\boldsymbol{j}-2\boldsymbol{k}-(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = -\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$. Then a normal to the plane is the cross product of these: $(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})\times(-\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}) = -\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$. [You should get either this vector or its negative no matter how you subtracted points; don't forget always to check that the cross product you get is perpendicular to the vectors you started with by dotting and making sure that you get 0.] So the vector equation of the plane is

$$(\boldsymbol{r}-(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}))\cdot(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = 0; \text{ or } \boldsymbol{r}\cdot(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = 1,$$

which is in the required form [**5 marks**]. [Check that the three given points satisfy this equation!]

To find the distance of this plane from the origin, we need the component of the position vector of a[ny] point in the plane in the direction of $-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$; the size of the normal is $\sqrt{3}$, so the distance is

$$(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/\sqrt{3} = 1/\sqrt{3} = \tfrac{1}{3}\sqrt{3} \approx 0.58$$

[**3 marks**].

We have just found a normal to the plane; a normal to the xy-plane is $\boldsymbol{k}$, so, as in Q1, the angle between the two planes is $\cos^{-1}(1/\sqrt{3}) \approx 0.96$ radians, 55 degrees [**2 marks**; total 10].

4. Differentiating

$$\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i}+\cos(t)\,\boldsymbol{j}-t^2\boldsymbol{k}$$

with respect to time, the particle's velocity is

$$\boldsymbol{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = \cos(t)\,\boldsymbol{i}-\sin(t)\,\boldsymbol{j}-2t\boldsymbol{k}$$

[**4 marks**, a gift]; its speed is

$$s = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{\cos^2(t)+\sin^2(t)+4t^2} = \sqrt{1+4t^2}$$

[**3 marks**; and its acceleration is

$$\boldsymbol{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t) = -\sin(t)\,\boldsymbol{i}-\cos(t)\,\boldsymbol{j}-2\boldsymbol{k}$$

[**3 marks**; total 10; grand total 40].