HG1M02—Applied Algebra for Engineers

Coursework 3—Solutions

1. *(a)*

$$\nabla f = \frac{\partial}{\partial x}(x^3y^3z^3)\boldsymbol{i} + \frac{\partial}{\partial y}(x^3y^3z^3)\boldsymbol{j} + \frac{\partial}{\partial z}(x^3y^3z^3)\boldsymbol{k} = $$
$$= 3x^2y^3z^3\boldsymbol{i} + 3x^3y^2z^3\boldsymbol{j} + 3x^3y^3z^2\boldsymbol{k}$$

[possible but not necessary to tidy that result up a little];

(b)

$$\nabla\cdot\nabla f = \nabla\cdot(3x^2y^3z^3\boldsymbol{i} + 3x^3y^2z^3\boldsymbol{j} + 3x^3y^3z^2\boldsymbol{k}) = $$
$$= \frac{\partial}{\partial x}(3x^2y^3z^3) + \frac{\partial}{\partial y}(3x^3y^2z^3) + \frac{\partial}{\partial z}(3x^3y^3z^2) = $$
$$= 6xy^3z^3 + 6x^3yz^3 + 6x^3y^3z$$

[same comment!];

(c) $$\nabla\cdot\boldsymbol{v} = \frac{\partial}{\partial x}(x^3y^2z) + \frac{\partial}{\partial y}(y^3z^2x) + \frac{\partial}{\partial z}(z^3x^2y) = 3x^2y^2z + 3y^2z^2x + 3z^2x^2y;$$

(d)

$$\nabla\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3y^2z & y^3z^2x & z^3x^2y \end{vmatrix} =$$

$$= \boldsymbol{i}\left(\frac{\partial}{\partial y}(z^3x^2y) - \frac{\partial}{\partial z}(y^3z^2x)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(z^3x^2y) - \frac{\partial}{\partial z}(x^3y^2z)\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(y^3z^2x) - \frac{\partial}{\partial y}(x^3y^2z)\right) =$$

$$= (z^3x^2 - 2y^3zx)\boldsymbol{i} + (x^3y^2 - 2z^3xy)\boldsymbol{j} + (y^3z^2 - 2x^3yz)\boldsymbol{k}.$$

[Care needed with signs!]

[3 marks for each part, total 12.]

2.

$$A^2 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 4 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 2 & 9 \\ -4 & 2 & 1 \\ 22 & 7 & 23 \end{pmatrix};$$

$$M = A^2 - 4A - 10I = \begin{pmatrix} 11-4-10 & 2-4 & 9-8 \\ -4-8 & 2+4-10 & 1+4 \\ 22-16 & 7-4 & 23-16-10 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 1 \\ -12 & -4 & 5 \\ 6 & 3 & -3 \end{pmatrix};$$

$$MA = \begin{pmatrix} -3 & -2 & 1 \\ -12 & -4 & 5 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 4 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = -3I.$$

Since $MA = -3I$, $(-\frac{1}{3}M)A = I$, and so

$$A^{-1} = -\tfrac{1}{3}M = \begin{pmatrix} 1 & 2/3 & -1/3 \\ 4 & 4/3 & -5/3 \\ -2 & -1 & 1 \end{pmatrix}.$$

[Worth noting: The characteristic equation of A is [check!] $\lambda^3 - 4\lambda^2 - 10\lambda + 3 = 0$; or, in other words, re-arranging slightly, $(\lambda^2 - 4\lambda - 10)\times\lambda = -3$. The similarity with the result $MA = -3I$ is not accidental.]

[3 marks each for A^2, M, MA and A^{-1}, total 12.]

3. There are many different possible orders, but your tableau should look *something* like:

	1	1	2	2	... (1)
	2	–1	–1	1	... (2)
	4	1	4	3	... (3)
(2)–2×(1)	0	–3	–5	–3	... (4)
(3)–4×(1)	0	–3	–4	–5	... (5)
(5)–(4)	0	0	1	–2	... (6)
(5)+4×(6)	0	–3	0	–13	... (7)
(7)÷–3	0	1	0	13/3	... (8)
(1)–(8)–2×(6)	1	0	0	5/3	... (9)

and from equations (9), (8) and (6) we can read off $x = \frac{5}{3}$, $y = \frac{13}{3}$, $z = -2$.

[5 marks for the tableau, 3 for the solution, total 8.]

4. The equations of Q3 take the form $A\boldsymbol{x} = \boldsymbol{b}$, and so $MA\boldsymbol{x} = -3I\boldsymbol{x} = -3\boldsymbol{x} = M\boldsymbol{b}$; that is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} -3 & -2 & 1 \\ -12 & -4 & 5 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} -5 \\ -13 \\ 6 \end{pmatrix}.$$

So $x = \frac{5}{3}$, $y = \frac{13}{3}$, $z = -2$, as before.

[4 marks for the explanation, 4 for the calculation, total 8, grand total 40.]