HG1M02—Exam 2007—Solutions and Feedback

Section A

1 Expanding by the first row, the determinant is

$$1\times(3\times1-(-1)\times(-1))-(-1)\times((-1)\times1-(-1)\times2)+2\times((-1)\times(-1)-3\times2) =$$
$$= (3-1)+(-1+2)+2\,(1-6) = 2+1-10 = -7.$$

Alternatively, if we take $C'_2 = C_2+C_1$, $C'_3 = C_3-2C_1$, then we get two 0's in the first row, and the determinant is

$$1\times\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = 2\times(-3)-1\times1 = -7.$$

Other answers from adding the middle minor or otherwise getting signs wrong.
Ans: **C**.

2 Length is $\sqrt{1^2+(-2)^2+3^2} = \sqrt{14}$.
Ans: **D**.

3 The dot product of the two vectors is $-2+2-2 = -2$; to find the component, we must divide this by the size of the second vector, which is $\sqrt{4+4+1} = 3$. Other answers from multiplying or forgetting.
Ans: **B**.

4 *Ans*: **A**.

5 *Ans*: **D**.
Other answers are the dot product and various sign errors.

6 $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{k}$, and $\boldsymbol{k}\cdot\boldsymbol{c} = 1$. Or work out a [simple] 3×3 determinant.
Ans: **C**.

7 *Ans*: **B**.

8 A^T is 3×2, so A^TB is conformable for multiplication. [And so is B^TA.]
Ans: **C**.

9 The given matrix *is* symmetric, its determinant is zero so it has no inverse, because it is symmetric it equals its transpose; but it is *not* orthogonal.
Ans: **D**.

10 We find t_{32} from the third row of R and the second column of S; so it is $5\times7+2\times3+1\times2 = 43$. Other answers from assorted rows and columns.
Ans: **C**.

11 *Ans*: **A**.
You can just multiply out to see which one works, or write down the 2×2 matrix of co-factors and divide by the determinant, which is $1\times3-2\times3 = -3$.

12 The equation for the eigenvalues is

$$(2-\lambda)(3-\lambda)-2\times6 = 0,$$
$$\text{or } \lambda^2-5\lambda-6 = 0,$$

which has roots $\lambda = 6$ and $\lambda = -1$.
Ans: **B**.

Comments: Section A was very well done; the average performance was about 9 or 10 answers right, 2 or 3 wrong or 'abstain'.

Section B

13 (a) A typical point on the cable starting at A is

$$(0,0,5)+s\times(5,4,1),$$

and on the cable starting at B is

$$(0,3,0)+t\times((4,-1,h)-(0,3,0)) = (0,3,0)+t\times(4,-4,h).$$

[4]

The cables will intersect if there are s and t such that

$$(0,0,5)+s\times(5,4,1) = (0,3,0)+t\times(4,-4,h).$$

That is, $5s = 4t$, $4s = 3-4t$, and $5+s = ht$. [2]

From the first two equations, $4t = 5s = 3-4s$, so $s = \frac{1}{3}$, $t = \frac{5}{4}s = \frac{5}{12}$, and so $h = (5+s)/t = \frac{16}{3}/\frac{5}{12} = \frac{64}{5} = 12.8$, as required. [2]

(b) So a vector normal to both cables is

$$\boldsymbol{v} = (5,4,1)\times(4,-4,8) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 5 & 4 & 1 \\ 4 & -4 & 8 \end{vmatrix} = (32+4)\boldsymbol{i}-(40-4)\boldsymbol{j}+(-20-16)\boldsymbol{k}.$$

We can take $\boldsymbol{v} = 36\boldsymbol{i}-36\boldsymbol{j}-36\boldsymbol{k}$. [Or just $\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$.] [5]

$\overrightarrow{AB} = (0,3,-5)$, so its component in the direction of $\boldsymbol{v}$ is

$$\overrightarrow{AB}\cdot\boldsymbol{v}/|\boldsymbol{v}| = (0,3,-5)\cdot(1,-1,-1)/\sqrt{3} = 2/\sqrt{3}$$

[$= \frac{2}{3}\sqrt{3} \approx 1.15$]. [4]

(c) The distance just found is the perpendicular separation of parallel planes each containing one of the cables, so is also the closest approach of the lines of the cables. So the closest approach is [at least] ≈ 1.15, and the cables are safely separated. [3]

Comments: Only a few students seriously attempted this question; many of you just wrote down the equations of the lines and then gave up. Those who persisted often got it right, and seemed to find the question quite easy.

14 (a) The velocity of the particle is

$$\boldsymbol{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = -\sin(t)\,\boldsymbol{i} + \cos(t)\,\boldsymbol{j} + 2\boldsymbol{k},$$ [3]

and so the speed is

$$\sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2} = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}.$$ [2]

The acceleration is then

$$\boldsymbol{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t) = -\cos(t)\,\boldsymbol{i} - \sin(t)\,\boldsymbol{j} + 0\boldsymbol{k},$$ [2]

which has magnitude $\sqrt{(-\cos t)^2 + (\sin t)^2} = 1$. As the force $\boldsymbol{F}$ acting on the particle satisfies $\boldsymbol{F} = m\boldsymbol{a}$, the magnitude of $\boldsymbol{F}$ is m, as required. [1]
Also $\boldsymbol{r} + \boldsymbol{a} = 2t\,\boldsymbol{k}$, or $(0, 0, 2t) - \boldsymbol{r} = \boldsymbol{a} = \boldsymbol{F}/m$, showing that $\boldsymbol{F}$ acts through the point $(0, 0, 2t)$. [2]

(b) If

$$T(x, y, z) = x^2 y \sin z,$$

then

$$\boldsymbol{u} = \nabla T = \frac{\partial T}{\partial x}\boldsymbol{i} + \frac{\partial T}{\partial y}\boldsymbol{j} + \frac{\partial T}{\partial z}\boldsymbol{k} = 2xy \sin z\,\boldsymbol{i} + x^2 \sin z\,\boldsymbol{j} + x^2 y \cos z\,\boldsymbol{k}.$$ [4]

Also,

$$\nabla\cdot\boldsymbol{u} = \frac{\partial}{\partial x}2xy \sin z + \frac{\partial}{\partial y}x^2 \sin z + \frac{\partial}{\partial z}x^2 y \cos z = 2y \sin z + 0 - x^2 y \sin z = (2 - x^2) y \sin z,$$ [3]

and

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy \sin z & x^2 \sin z & x^2 y \cos z \end{vmatrix}$$
$$= (x^2 \cos z - x^2 \cos z)\boldsymbol{i} - (2xy \cos z - 2xy \cos z)\boldsymbol{j} + (2x \sin z - 2x \sin z)\boldsymbol{k} = \boldsymbol{0},$$

as required. [3]

Comments: Common mistakes were to get the signs of the derivatives of sine and cosine wrong, and to confuse vectors and scalars. Perhaps surprisingly, $\nabla\times\boldsymbol{u}$ was much better done than either $\nabla\cdot\boldsymbol{u}$ or finding $\boldsymbol{v}$; almost everyone knew the answer would be a vector, and taking care with signs in the determinant perhaps helped with the signs attached to trig functions. You lost serious marks if you thought that $\nabla\cdot\boldsymbol{u}$ was a vector or that ∇T was a scalar.

Scarcely anyone seemed to understand what was required for the last part of (a); the idea was that the force $\boldsymbol{F}$ was acting at $\boldsymbol{r}$, so you had to show that the line through $\boldsymbol{r}$ in the direction of $\boldsymbol{F}$ went through the given point. [Mechanically, $\boldsymbol{r}(t)$ describes a particle spiralling round and up the z-axis, from which you might have some intuitive idea of what forces are needed to make the particle do this.]

But marks of 15/20 or better were very common, so I thought the class as a whole did well.

15 (a) There is a non-trivial solution if the determinant

$$\begin{vmatrix} 2 & 1 & t \\ 1 & 4 & 1 \\ 3 & -1 & -10 \end{vmatrix} = 0.$$

[2]

Expanding by the top row, this is

$$2\times(-40+1)-1\times(-10-3)+t\times(-1-12) = -78+13-13t = -65-13t = 0,$$

so $t = -65/13 = -5$. [2]

Arbitrarily taking $z = 1$, we have $2x+y-5 = 0$ and $x+4y+1 = 0$; so [first equation minus twice the second] $0x-7y-7 = 0$, so $y = -1$ and $x = 3$. So a non-trivial solution is $(x, y, z) = (3, -1, 1)$. [3]

(b) The matrix form is

$$\begin{pmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ -2 \end{pmatrix},$$

[2]

with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right).$$

[2]

$$\begin{array}{rrr|rl} 1 & -4 & -2 & 21 & (1) \\ 2 & 1 & 2 & 3 & (2) \\ 3 & 2 & -1 & -2 & (3) \\ 0 & 9 & 6 & -39 & (4) = (2)-2\times(1) \\ 0 & 14 & 5 & -65 & (5) = (3)-3\times(1) \\ 0 & 3 & 2 & -13 & (6) = (4)/3 \\ 0 & -1 & -5 & 0 & (7) = (5)-5\times(6) \\ 0 & 0 & -13 & -13 & (8) = (6)+3\times(7) \end{array}$$

[(6) and (7) not really needed, but simplify the arithmetic]. [5]

From (8), we read off $-13z = -13$ or $z = 1$; so (6) becomes $3y+2z = -13$ or $y = (-13-2)/3 = -5$; and (1) becomes $x-4y-2z = 21$ or $x = 21-20+2 = 3$. [3]

So the solution is $(x, y, z) = (3, -5, 1)$. [1]

Comments: Signs, signs, signs! There were a lot of perfectly correct solutions, but also surprisingly many who told me that $65+13t = 0$ and so $t = 5$, or that $13t = -78-(-13) = -91$. Likewise, in (b), subtracting negative numbers while doing the row operations often led to mistakes, after which the solutions went off into la-la land. *Hint*: If you have a problem that involves only small whole numbers, it's worth working out the determinant of the matrix; basically, that will then be the denominator in any fractions that result. If you then get make an arithmetic slip, there is a good chance that you will get completely the wrong sorts of fraction, and the error will be obvious.

Practically everyone showed that they understood all the concepts behind this question, so marks were very good.

16 The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0;$$

[3]

expanding by the first column, we have

$$(1-\lambda)[(2-\lambda)(-1-\lambda)-1]+[-1-\lambda+2] = 0,$$

[3]

and so $(1-\lambda)(2-\lambda)(-1-\lambda) = 0$, and the eigenvalues are 1, 2 and -1. [2]

We have

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+3-2 \\ -1+6+1 \\ 0+3-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 2\times\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix},$$

confirming that $(1,3,1)^T$ is an eigenvector corresponding to the eigenvalue 2. [OK to find an eigenvector directly, as below, instead of verifying.] [4]

If $(x,y,z)^T$ is an eigenvector correponding to the eigenvalue 1, we must have

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y-2z \\ -x+2y+z \\ 0x+y-z \end{pmatrix} = 1\times\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and so $y = 2z$, $x = y+z$ and $y = 2z$ [redundant!] respectively; taking, arbitrarily, $z = 1$, we have $y = 2$, $x = 3$, so an eigenvector is $(3,2,1)^T$. [4]

Similarly, if $(x,y,z)^T$ is an eigenvector correponding to the eigenvalue -1, we must have

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y-2z \\ -x+2y+z \\ 0x+y-z \end{pmatrix} = -1\times\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and so $2x+y-2z=0$, $x = 3y+z$ and $y = 0$ respectively; taking, arbitrarily, $z = 1$, we have $y = 0$, $x = 1$, so an eigenvector is $(1,0,1)^T$. [4]

Comments: Very many correct solutions to this question. Practically everyone understood how to find eigenvalues and eigenvectors; there were just a few silly slips.

End