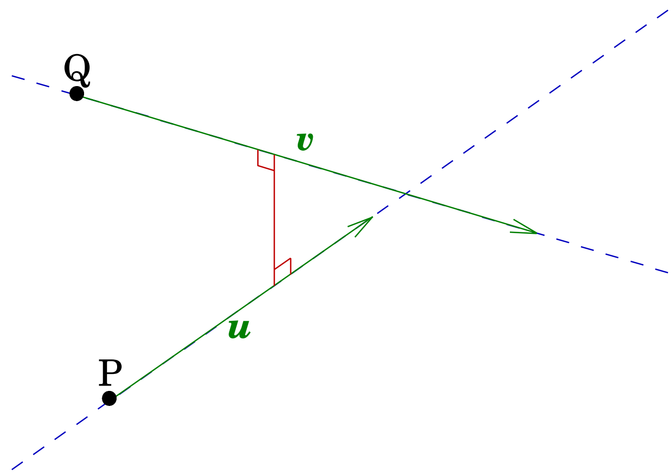
Distance between lines

Given two *skew* lines [that are neither parallel nor intersect], we may want to know how far apart they are. The nearest approach between the lines occurs between points, one on each line, such that the line joining them is perpendicular to both lines:



Given the positions of points P and Q and vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ along the lines, then there are three possible ways of finding this distance.

- A typical point along the first line is $\boldsymbol{p}+t\boldsymbol{u}$, where $\boldsymbol{p}$ is the position vector of P and t is a scalar; and similarly on the second line is $\boldsymbol{q}+s\boldsymbol{v}$, where $\boldsymbol{q}$ is the position vector of Q and s is a scalar. The distance between these points is

$$\sqrt{(\boldsymbol{p}-\boldsymbol{q}+t\boldsymbol{u}-s\boldsymbol{v})\cdot(\boldsymbol{p}-\boldsymbol{q}+t\boldsymbol{u}-s\boldsymbol{v})},$$

  which is a somewhat messy function of s and t, which we can differentiate with respect to s and t, to find its minimum. Ugh. Not recommended.

- We can directly use the fact that we want the mutual perpendicular. So we want

$$(\boldsymbol{p}-\boldsymbol{q}+t\boldsymbol{u}-s\boldsymbol{v})\cdot\boldsymbol{u} = (\boldsymbol{p}-\boldsymbol{q}+t\boldsymbol{u}-s\boldsymbol{v})\cdot\boldsymbol{v} = 0,$$

giving two equations for s and t. If we need to find the actual position of the minimum distance, this is quite a good method. But very commonly, …

- … we only want to know the distance, not where it happens. If so, then the distance apart is the component of *any* line joining points in the two lines—for example, the line PQ—in the direction of the common normal. But we know that direction: it is the direction of $\boldsymbol{u}\times\boldsymbol{v}$.

- **Example:** If the points A, B, C and D are at $(1,1,1)$, $(4,-1,2)$, $(1,-1,-1)$ and $(5,2,0)$ respectively, what is the distance between the lines AB and CD?

$\overrightarrow{AB} = (4,-1,2)-(1,1,1) = (3,-2,1)$; $\overrightarrow{CD} = (5,2,0)-(1,-1,-1) = (4,3,1)$; so a common normal vector is

$$\overrightarrow{AB}\times\overrightarrow{CD} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -5\boldsymbol{i}+\boldsymbol{j}+17\boldsymbol{k}.$$

Check: this is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$.
We want the component of $\overrightarrow{AC} = (1,-1,-1)-(1,1,1) = (0,-2,-2)$ in this direction; that is

$$(0,-2,-2)\cdot(-5,1,17)/|(-5,1,17)| =$$
$$(0-2-34)/\sqrt{25+1+289} = -36/\sqrt{315} \approx -2.03.$$

So the required distance is about 2.03 units.
Check: We would get the same distance from any pair of points on the two lines.

Vector Calculus

Differentiation of a vector by a scalar

Nothing really new here. Typical example—the scalar is t, the time. In this case, if a particle is at some position $\boldsymbol{r}(t)$—that is, $\boldsymbol{r}$ depends on t, so the particle is at different positions at different times—then its velocity $\boldsymbol{v}$ is $\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ and its acceleration $\boldsymbol{a}$ is $\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\left[=\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right]$. Differentiation with respect to time is often written by using a dot, so we could equally write $\boldsymbol{v} = \dot{\boldsymbol{r}}$, $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$.

- **Example**: At time t, a particle is at $\boldsymbol{r} = t\boldsymbol{i}+t^2\boldsymbol{j}+t^3\boldsymbol{k}$. Find its position, velocity, speed and acceleration when $t = 0$ and when $t = 2$.

  Its velocity and acceleration are

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = 1\boldsymbol{i}+2t\boldsymbol{j}+3t^2\boldsymbol{k};$$
$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 0\boldsymbol{i}+2\boldsymbol{j}+6t\boldsymbol{k}.$$

  while the speed is

$$s = |\boldsymbol{v}| = \sqrt{1^2+(2t)^2+(3t^2)^2} = \sqrt{1+4t^2+9t^4}.$$

  Substituting,
  at $t = 0$, $\boldsymbol{r} = 0$, $\boldsymbol{v} = \boldsymbol{i}$, $s = 1$ and $\boldsymbol{a} = 2\boldsymbol{j}$; and
  at $t = 2$, $\boldsymbol{r} = 2\boldsymbol{i}+4\boldsymbol{j}+8\boldsymbol{k}$, $\boldsymbol{v} = \boldsymbol{i}+4\boldsymbol{j}+12\boldsymbol{k}$, $s = \sqrt{1+16+144} = \sqrt{161}$ and $\boldsymbol{a} = 2\boldsymbol{j}+12\boldsymbol{k}$.

By Newton's second Law, the force acting on a body is proportional to the acceleration, $\boldsymbol{F} = m\boldsymbol{a}$, where m is the mass, usually taken to be constant. So we can find the force when we know the acceleration: the particle in the previous example is subject to the force

$$\boldsymbol{F} = m\boldsymbol{a} = m(2\boldsymbol{j} + 6t\boldsymbol{k}).$$

In practical applications, we usually know $\boldsymbol{F}$ as a function of $\boldsymbol{r}$ ['if the body is *here*, then the forces on it will be *such-and-such*'], giving us a differential equation,

$$\boldsymbol{F}(\boldsymbol{r}) = m\boldsymbol{a} = m\ddot{\boldsymbol{r}},$$

for $\boldsymbol{r}(t)$. *Solving* this sort of equation is beyond the scope of this module! [But very important in mechanics.]

Spatial derivatives

Suppose we have a scalar or vector *field*. This means a scalar or vector function of position that is defined over some region of space—for example, the temperature or density or pressure [scalars] at each point, or the wind velocity or magnetic field or gravitational force [vectors] at each point. Then we can ask how that field changes with position. This usually involves *partial derivatives* [as discussed in HG1M01], because the field usually depends on x, y and z [or other co-ordinate system], as well as perhaps on t.

- **Example**: Suppose the pressure, p, in a gas varies according to the law $p = x^2 y e^{-z}$. Then it's natural to ask how p is changing with x, y and z. We have

  $$\frac{\partial p}{\partial x} = 2xye^{-z}; \quad \frac{\partial p}{\partial y} = x^2 e^{-z}; \quad \text{and} \quad \frac{\partial p}{\partial z} = -x^2 y e^{-z}.$$

  It's also natural to bundle these three 'components' into a vector, called the *gradient* of p, written 'grad p' or ∇p, and pronounced 'grad p' or 'del p':

  $$\nabla p = 2xye^{-z}\boldsymbol{i} + x^2 e^{-z}\boldsymbol{j} - x^2 y e^{-z}\boldsymbol{k}.$$

More generally,

$$\nabla p = \frac{\partial p}{\partial x}\boldsymbol{i} + \frac{\partial p}{\partial y}\boldsymbol{j} + \frac{\partial p}{\partial z}\boldsymbol{k} = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right).$$

We can 'factorise out' the p from this result, to give an 'operational' definition of ∇:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

∇p tells us how p is changing with position; its component in any direction tells us how fast p is increasing in that direction.