Divergence and Curl

We have just seen that ∇ looks rather like the vector

$$\nabla \;=\; \frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k} \;=\; \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right).$$

If we pretend that ∇ really is a vector, then it becomes natural to ask what happens if we use our vector algebra with it. This gives us two new spatial derivatives:

$$\nabla\cdot\boldsymbol{v} \text{ and } \nabla\times\boldsymbol{v},$$

where $\boldsymbol{v}$ is some vector field [such as the wind velocity at each point], and also combinations such as

$$\nabla\cdot\nabla p = \nabla^2 p;\;\; \nabla\times\nabla p \text{ and } \nabla\times\nabla\times\boldsymbol{v}.$$

$\nabla\cdot\boldsymbol{v}$ is called the *divergence* of $\boldsymbol{v}$, or ‘div $\boldsymbol{v}$’ or ‘del dot $\boldsymbol{v}$’; it is sometimes written out as ‘div $\boldsymbol{v}$’ rather than $\nabla\cdot\boldsymbol{v}$. $\nabla\times\boldsymbol{v}$ is called the *curl*, or sometimes the *rotation*, of $\boldsymbol{v}$, or ‘del cross $\boldsymbol{v}$’, and sometimes written out as ‘curl $\boldsymbol{v}$’, or occasionally ‘rot $\boldsymbol{v}$’ rather than $\nabla\times\boldsymbol{v}$. $\nabla^2 p$ is usually called ‘del squared p’, but also sometimes the *Laplacian* of p.

These operators and their combinations all have important physical properties, and if you look at advanced maths or engineering or physics textbooks you will often see ∇ in equations all over the place.

Divergence of a vector field

Given a vector field

$$\boldsymbol{v} = (v_x, v_y, v_z),$$

its divergence is

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

- **Example**: The wind velocity at $\boldsymbol{r} = (x, y, z)$ is

$$\boldsymbol{v}(\boldsymbol{r}) = \mathrm{e}^{-z}\cos(x+y)\boldsymbol{i} + \mathrm{e}^{-z}\sin(x-y)\boldsymbol{j}.$$

What is $\nabla \cdot \boldsymbol{v}$?

Here $v_x = \mathrm{e}^{-z}\cos(x+y)$, $v_y = \mathrm{e}^{-z}\sin(x-y)$ and $v_z = 0$.
So $\dfrac{\partial v_x}{\partial x} = -\mathrm{e}^{-z}\sin(x+y)$, $\dfrac{\partial v_y}{\partial y} = -\mathrm{e}^{-z}\cos(x-y)$ and $\dfrac{\partial v_z}{\partial z} = 0$.

Remember that when we do $\partial/\partial x$, we 'pretend' that y and z are constant, so you can think of v_x here as $\alpha\cos(x+\beta)$ with α and β constant, if that helps you. Similarly for $\partial/\partial y$ and v_y—take care with signs here, as y appears negatively in the formula for v_y.

So $\nabla \cdot \boldsymbol{v} = -\mathrm{e}^{-z}(\sin(x+y) + \cos(x-y))$.

$\dfrac{\partial v_x}{\partial x}$ tells us how rapidly the x-component of $\boldsymbol{v}$ is increasing in the x-direction, so how rapidly the wind [in this case] is becoming 'spaced out' in the x-direction. Similarly for the y- and z-directions; the sum, $\nabla \cdot \boldsymbol{v}$, is telling us how rapidly the wind is becoming 'spaced out' in total. An incompressible fluid, such as water, cannot be 'spaced out' at all, so its divergence is zero, $\nabla \cdot \boldsymbol{v} = 0$. Air can become 'spaced out', $\nabla \cdot \boldsymbol{v} > 0$, if its density is decreasing, or can become compressed, $\nabla \cdot \boldsymbol{v} < 0$, if its density is increasing. An incompressible field, $\nabla \cdot \boldsymbol{v} = 0$ is also said to be *solenoidal*. Electricity and gravity are solenoidal *except* at points where electric charges and massive particles 'create' electric and gravity fields.

- **Example**: For what value of γ is the velocity field $\boldsymbol{v}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + \gamma z\boldsymbol{k}$ a possible description of the flow of water?

  We need the flow to satisfy $\nabla \cdot \boldsymbol{v} = 0$. But

  $$\nabla \cdot \boldsymbol{v} \;=\; \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial}{\partial x}\gamma z \;=\; 1 + 1 + \gamma,$$

  so we need $\gamma = -2$.

Curl of a vector field

Given a vector field

$$\boldsymbol{v} = (v_x, v_y, v_z),$$

its curl is

$$\operatorname{curl}\boldsymbol{v} = \nabla\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}.$$

- **Example**: The wind velocity at $\boldsymbol{r} = (x, y, z)$ is still

$$\boldsymbol{v}(\boldsymbol{r}) = \mathrm{e}^{-z}\cos(x+y)\boldsymbol{i} + \mathrm{e}^{-z}\sin(x-y)\boldsymbol{j}.$$

What is $\nabla\times\boldsymbol{v}$?

Here $v_x = \mathrm{e}^{-z}\cos(x+y)$, $v_y = \mathrm{e}^{-z}\sin(x-y)$ and $v_z = 0$. So we want

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{e}^{-z}\cos(x+y) & \mathrm{e}^{-z}\sin(x-y) & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\mathrm{e}^{-z}\sin(x-y)\boldsymbol{i} + \frac{\partial}{\partial z}\mathrm{e}^{-z}\cos(x+y)\boldsymbol{j} + \left(\frac{\partial}{\partial x}\mathrm{e}^{-z}\sin(x-y) - \frac{\partial}{\partial y}\mathrm{e}^{-z}\cos(x+y)\right)\boldsymbol{k}$$

$$= \mathrm{e}^{-z}\sin(x-y)\boldsymbol{i} - \mathrm{e}^{-z}\cos(x+y)\boldsymbol{j} + \left(\mathrm{e}^{-z}\cos(x-y) + \mathrm{e}^{-z}\sin(x+y)\right)\boldsymbol{k}.$$

Messy, like all cross products, but not that *difficult*!

Not examinable!

The curl of a vector field represents the local *angular velocity* or *vorticity* of a field. It tells us how much the field is 'rotating'. Many fields of interest are *irrotational*; that is, they satisfy $\nabla\times\boldsymbol{v} = \mathbf{0}$. In many others, the vorticity is 'conserved'; if you make some water swirl [eg in the bath], then you can see the vortices moving around, and once you have set them up, they show a strong tendency to persist. The vortices are quite like 'particles'. When they are set up by aircraft, they are strong and invisible, and a serious danger to following aircraft that happen to bump into them. In other fields, the vorticity is related to other quantities; for example, a rotating electric field generates magnetism, and vice versa.

We do not pursue these issues in this module, but Laplacians occur whenever there is 'potential' [potential energy, electric potential], and also in rotations of rotations. Many applications lead either to a zero Laplacian, or to one which is proportional to the acceleration [the 'wave equation']—it was spotting this in the equations for electromagnetism that led to the realisation that light waves are an electromagnetic phenomenon, and that electromagnetic waves, moving at the speed of light, 'radio waves' were possible.

Some of the vector algebra carries over to combinations; for example, $\nabla\cdot\nabla\times\boldsymbol{v} = 0$ and $\nabla\times\nabla p = \mathbf{0}$; curls are solenoidal [or vortices are incompressible!], and gradients are irrotational. But be careful with these ideas!