Matrix multiplication

This is a somewhat weird process, but it's motivated by the combinations of matrices that arise naturally in practical use. Suppose that A is an $m{\times}n$ matrix, and B is a $p{\times}q$ matrix. Then we can multiply A by B to form the matrix AB if [and *only* if] $n = p$. If so, the matrices are *conformable* for multiplication, and AB is an $m{\times}q$ matrix. In 'short-hand', $(m{\times}n){\times}(n{\times}q)$ is $(m{\times}q)$, and the 'middle' sizes get removed.

To find the matrix AB, we do something even more strange [hold tight!]:

$$(ab)_{ij} = \sum_k a_{ik} b_{kj} \; [= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots].$$

Note that as k changes, a_{ik} picks out the different elements in the i-th row of A, and similarly b_{kj} picks out the corresponding elements in the j-th column of B. They *will* correspond, because the test for conformability ensures that rows of A and columns of B have the same size.

If we think of the i-th row of A as the row vector $\boldsymbol{A}_i$, and the j-th column of B as the column vector $\boldsymbol{B}_j$, then what we have found is just the dot, or inner product, $\boldsymbol{A}_i \cdot \boldsymbol{B}_j$ of the two vectors—exactly as before if A has three columns, but an obvious generalisation otherwise. [Vector algebra took the phrase 'inner product' from the matrix version.]

So AB is the array of inner products of rows of A with columns of B. Note the asymmetry. The inner products of columns of A with rows of B may not exist [even if AB exists, the columns of A may have different lengths from the rows of B]; if BA does also exist, it may be a different size from AB; and if it happens to be the same size, it will usually have different elements. If the elements happen to be the same, A and B are said to *commute*; this is a useful property! But in general, we have to be very careful when doing algebra with matrices to keep products in the right order.

Examples

- Given the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix};\ B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 0 & 3 \end{pmatrix}$$

[as in the previous lecture!], which of AB, AC, A^2, CA, C^2, CC^T and C^TA^T exist? Find those which do.

So A, B and A^T are 2×2, C is 2×3, and C^T is 3×2. So AB, AC, A^2, CC^T and C^TA^T all exist [middle sizes 2, 2, 2, 3 and 2], while CA and C^2 do not [middle sizes 3/=2 in both cases].

So [doing the first one in full, you should check the others!]:

$$AB = \begin{pmatrix} 1\times1+0\times(-1) & 1\times1+0\times2 \\ 2\times1+4\times(-1) & 2\times1+4\times2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 10 \end{pmatrix};$$

$$AC = \begin{pmatrix} 1 & -1 & 2 \\ 22 & -2 & 16 \end{pmatrix};\quad A^2 = \begin{pmatrix} 1 & 0 \\ 10 & 16 \end{pmatrix};\quad CC^T = \begin{pmatrix} 6 & 11 \\ 11 & 34 \end{pmatrix};$$

$$C^TA^T = \begin{pmatrix} 1 & 22 \\ -1 & -2 \\ -2 & 16 \end{pmatrix} = (AC)^T.$$

The last example is not an accident—it is generally true that if A and C are conformable for multiplication, then so are C^T and A^T *in that order* [swapped around!], and C^TA^T is the transpose of AC. Note that for any matrix M, M and M^T are conformable for multiplication, in either order, but MM^T and M^TM are the same size only if M is square, and are not usually equal.

- If A is as above, and I_2, O_{22} are the 2×2 identity and zero matrices, find I_2A, AI_2, $O_{22}A$ and AO_{22}.

$$I_2A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = A; \quad AI_2 = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = A;$$

$$O_{22}A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_{22;} \quad AO_{22} = O_{22}.$$

Generally, if I and O are the right sizes of identity/zero matrices, then for any matrix M, $IM = MI = M$ and $OM = MO = O$. [Note that if M is not square, this is an abuse of notation, because the Is and Os are not the same.]

- For the same matrices A and B as above, find BA.

$$BA = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 8 \end{pmatrix} \neq AB.$$

Never swap matrix products around unless you are *sure* that this is one of the exceptional cases, such as $AI = IA$, where the matrices commute.

But quite a lot of algebra works. For example, $(A+B)C = AC+BC$ [check!], and $A(BC) = (AB)C = ABC$ [check!] [for the above matrices, and in general *provided* the matrices are conformable].

Relation to simultaneous equations

Note that the simultaneous equations we started with, way back,

$$\begin{aligned} 3x + 2y &= 5, \\ 4x + 5y &= 6, \end{aligned}$$

may be written as a matrix equation:

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix},$$

which is in the form $M\boldsymbol{x} = \boldsymbol{b}$ where M is a 2×2 matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are 2×1 column vectors. Now note something rather clever. If we take N to be the matrix

$$N = \begin{pmatrix} \frac{5}{7} & -\frac{2}{7} \\ -\frac{4}{7} & \frac{3}{7} \end{pmatrix} = \tfrac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}$$

[we'll see later how to find N], then

$$NM = \tfrac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} = \tfrac{1}{7}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = I_2.$$

So, since $M\boldsymbol{x} = \boldsymbol{b}$,

$$NM\boldsymbol{x} = I_2\boldsymbol{x} = \boldsymbol{x} = N\boldsymbol{b} = \tfrac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \tfrac{1}{7}\begin{pmatrix} 13 \\ -2 \end{pmatrix},$$

so $x = 13/7$ and $y = -2/7$, as before. Neat?

Since $NM = I$, we write $N = M^{-1}$ and call N the *inverse* of M. Technically, N is the *left* inverse of M; and a matrix P such that $MP = I$ is the *right* inverse of M. Not proved here: If M is a square matrix, and $NM = I$, then $MN = I$ also, so the left and right inverses are the same. **Not all matrices, even if they are square, and even if they are non-zero, have inverses.**