Inverse matrices

If there is a matrix $N = M^{-1}$ such that $MN = NM = I$, then there are lots of nice algebraic things we can do. Just as we can't divide two vectors, similarly we can't divide two matrices. But we can do the next best thing: instead of dividing by M, we can multiply by M^{-1}. *Most* square matrices M do have inverses, and we can [but are not going to!] write them down formally in terms of lots of determinants—the determinant, written $\det M$, which has the same elements in the same positions as M, and the minors of that determinant [that is, the determinants obtained by striking out a particular row and column].

Square matrices that do *not* have an inverse are called *singular*. And so matrices that *do* have an inverse are *non-singular*. The formal inverse [which we're not going to look at in this module] shows that a matrix is singular if, and only if, its determinant is zero.

Why am I not going to show you the formal inverse? It's because it's of almost no practical use. To invert a $100{\times}100$ matrix, we need to work out a $100{\times}100$ determinant and its 10^4 $99{\times}99$ minors. Finding these is quite impracticable—*except* by finding the inverse efficiently and reading off the minors from it!

There is one exception. For a $2{\times}2$ matrix, we need a determinant of order 2 and its 4 minors. So, ….

Inverse of a 2×2 matrix

Given a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is $\det M = ad - bc$. *Provided that* $\det M \neq 0$, we can write down

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

You should check that with M^{-1} as just given, $MM^{-1} = M^{-1}M = I$.

- Note that a and d have swapped places, b and c have stayed where they are but changed sign. They have changed sign because of the 'chequer-board' pattern of determinants; they have stayed where they are because the pattern of minors is transposed [so they have swapped twice].

Examples

- Find the inverses, where possible, of

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix};\ B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix};\ C = \begin{pmatrix} 6 & -5 \\ -3 & 2 \end{pmatrix};\ D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \text{and } E = \begin{pmatrix} -\frac{2}{3} & -\frac{5}{3} \\ -1 & -2 \end{pmatrix}.$$

We note that $\det A = 3\times 1 - 2\times 1 = 1$, $\det B = 1\times 4 - 2\times 2 = 0$, $\det C = 6\times 2 - (-3)\times(-5) = -3$, $\det D = 1$ and $\det E = -\frac{2}{3}\times(-2) - (-\frac{5}{3})\times(-1) = -\frac{1}{3}$. So B is singular and has no inverse; but

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix};\ C^{-1} = \frac{1}{-3}\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{5}{3} \\ -1 & -2 \end{pmatrix}.$$

Finally, $D = I$, so $D^{-1} = I$, and $C^{-1} = E$, so $E^{-1} = C$. Check all these!

Special forms and properties of inverses

- We have already seen particular cases of *(a)* $I^{-1} = I$; and *(b)* if $M^{-1} = N$, then $N^{-1} = M$. Also *(c)* if $M^{-1} = N$, then $(M^T)^{-1} = (M^{-1})^T$ [the inverse of the transpose is the transpose of the inverse]; and *(d)* if A and B are conformable for multiplication and are non-singular, then $(AB)^{-1} = B^{-1}A^{-1}$ [the inverse of a product is the swapped-around product of the inverses]. [Check for the last result: $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. Note that we have to keep products of matrices in the right order when we do algebra with them!]

  Examples: using the matrices above,

  $$A^T = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \text{ so } (A^T)^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix},$$

  using the previous result. Also,

  $$AC = \begin{pmatrix} 12 & -11 \\ 3 & -3 \end{pmatrix} \text{ so } (AC)^{-1} = C^{-1}A^{-1} = -\tfrac{1}{3}\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} -3 & 11 \\ -3 & 12 \end{pmatrix}.$$

  For 2×2 matrices, these results are not much, if any, easier than writing down the inverse directly, but they work just as well for larger matrices.

- If a matrix is symmetric, or anti-symmetric, or upper triangular, or lower triangular, or diagonal, then so is its inverse [if one exists].

  These results are immediate from ‘the formula’ for 2×2 matrices, but they apply to any matrix.

Diagonal matrices

For example, if

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \text{ then } D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

A diagonal matrix has 'disentangled' a set of simultaneous equations; each equation includes only a single variable. So diagonal matrices are 'targets' of many matrix applications.

Orthogonal matrices

A matrix is *orthogonal* if its transpose is its inverse, $M^T = M^{-1}$, as for example $\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$. Orthogonal matrices represent rotations and reflexions, so they are very important in computer graphics and other applications where you want to turn something round without distortion or change of scale. [If you think of $\frac{1}{2}\sqrt{3} = \cos\frac{1}{6}\pi$, $\frac{1}{2} = \sin\frac{1}{6}\pi$ in the example, the connexion with rotations may be more obvious.]

Inverting a larger matrix

If we aren't going to do this by formula, how can we do it? We need to step back a little. See the sections on *Gauss elimination* and *Gauss–Jordan* in the booklet.