**Gaussian elimination and Gauss–Jordan**

[See the examples in the booklet.]

Note that the example given using variables $[w, x, y, z]$ is just a somewhat more complicated case of the simultaneous equations you solved at school/college, and that we solved formally when we talked about determinants. Gaussian elimination is just a further formalisation of that process.

The row operations involved [in the matrix formulation] are very similar to those we used in evaluating determinants. So if we stick to operations of the form $row_i = row_i \pm c \times row_j$, and don't swap rows or scale them, as in the *Gaussian elimination* example, we have virtually found $\det A$ as a by-product of the reduction of $A$ to the upper-triangular form $B$: $\det A = 2\times6\times3\times2 = 72$. This is a Good Way to find determinants of large matrices, eg by computer program. [If you do swap/scale rows, then the determinant equally changes sign or is scaled.]

The Gauss–Jordan process systematically then reduces the upper-triangular form, such as $B$, to an identity matrix, $I$, after which the solution is trivial. If we initially augment by an identity matrix, then the row-reduction of $A$ to $I$ similarly reduces $I$ to $A^{-1}$, so we can read off the inverse of $A$. This is a Good Way to find inverses. [Even of $2\times2$ matrices.]

For large matrices, this is far from the full story. Doing so many row operations allows rounding errors to accumulate unless you are careful. See later modules in *Numerical Analysis*. At least read up about *pivoting* and about *conditioning* before trying this in Real Life.

**Examples**

- Solve the equations

$$\begin{aligned} x+y-z &= 4 \\ 2x-y+3z &= -3 \\ -x-2y+2z &= -7 \end{aligned}$$

by using row reductions.

In matrix form, this is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -7 \end{pmatrix}.$$

As a Gauss–Jordan *tableau*, this looks like:

| | | | | | |
|---|---|---|---|---|---|
| | 1 | 1 | −1 | 4 | … (1) |
| | 2 | −1 | 3 | −3 | … (2) |
| | −1 | −2 | 2 | −7 | … (3) |
| (2)−2×(1) | 0 | −3 | 5 | −11 | … (4) |
| (3)+(1) | 0 | −1 | 1 | −3 | … (5) |
| (4)−3×(5) | 0 | 0 | 2 | −2 | … (6) |
| (6)÷ 2 | 0 | 0 | 1 | −1 | … (7) |
| (5)−(7) | 0 | −1 | 0 | −2 | … (8) |
| −(8) | 0 | 1 | 0 | 2 | … (9) |
| (1)−(9)+(7) | 1 | 0 | 0 | 1 | … (10) |

Equations (7), (9) and (10) [the 'back substitution' phase] tell us that $z = -1$, $y = 2$ and $x = 1$.

Equations (1), (4) and (6) correspond to the upper-triangular matrix obtained before back-substitution. The above is my recommended layout.

- Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -2 & 2 \end{pmatrix}$$

Putting a unit matrix into the tableau, we have:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | –1 | 1 | 0 | 0 | |
| 2 | –1 | 3 | 0 | 1 | 0 | |
| –1 | –2 | 2 | 0 | 0 | 1 | |
| 0 | –3 | 5 | –2 | 1 | 0 | |
| 0 | –1 | 1 | 1 | 0 | 1 | |
| 0 | 0 | 2 | –5 | 1 | –3 | |
| 0 | 0 | 1 | -5/2 | 1/2 | –3/2 | * |
| 0 | –1 | 0 | 7/2 | –1/2 | 5/2 | |
| 0 | 1 | 0 | –7/2 | 1/2 | –5/2 | * |
| 1 | 0 | 0 | 2 | 0 | 1 | * |

So, from the lines (*), we can read off the lines corresponding to a unit matrix to the left of the line:

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ -\frac{7}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

Note that the numbers in the left half of the tableau are *exactly* the same as in the previous example.

**What can go wrong?**

It looks as though the Gauss–Jordan process should always work; but I said earlier that not every matrix has an inverse. So how does the process fail? Let us tweak a previous example:

- Solve the equations

$$x+y-z = 4$$
$$2x-y+z = -3$$
$$-x-2y+2z = -7$$

by using row reductions. [The second equation previously was $2x-y+\mathbf{3}z = -3$.]

So the tableau starts off:

| | | | | | |
|---|---|---|---|---|---|
| | 1 | 1 | –1 | 4 | … (1) |
| | 2 | –1 | 1 | –3 | … (2) |
| | –1 | –2 | 2 | –7 | … (3) |
| (2)–2×(1) | 0 | –3 | 3 | –11 | … (4) |
| (3)+(1) | 0 | –1 | 1 | –3 | … (5) |
| (4)–3×(5) | 0 | 0 | 0 | –2 | … (6) |

Oops! Equation (6) now says $0x+0y+0z = -2$; the equations are inconsistent. There is no solution, and no way to complete the Gauss–Jordan process, either to solve the equations, or to invert the matrix.

Similarly, if our second equation had been $2x-y+z = -1$, then equation (6) would have been $0x+0y+0z = 0$ [check!]. This would not have been inconsistent; but it is also no use to us. Effectively we have no equation for $z$: we can choose $z$ arbitrarily, and then back-substitute to find $y$ and then $x$ in terms of $z$. There is a solution for each possible value of $z$; but still no way to invert the matrix.