Homogeneous equations

A set of linear equations is *homogeneous* if the constant terms on the right-hand side are all zero; that is, the equations take the form

$$M\boldsymbol{x} = \mathbf{0}.$$

If the matrix M is non-singular, then, by definition, it has an inverse matrix M^{-1} such that $M^{-1}M = I$, and so

$$\boldsymbol{x} = I\boldsymbol{x} = M^{-1}M\boldsymbol{x} = M^{-1}\mathbf{0} = \mathbf{0}.$$

In other words, $\boldsymbol{x} = \mathbf{0}$ is a solution of the equations, and what is more is the *only* solution of the equations. This is the *trivial* solution.

Homogeneous equations always have the trivial solution. To be ‘interesting’, they have to have non-trivial solutions, and this can happen only if the corresponding matrix, M, is *singular*.

As we have seen, for a square matrix M, this will happen if and only if the Gauss–Jordan process breaks down, which in turn will happen if and only if M has a zero determinant.

- *Example*: Solve the equations $M\boldsymbol{x} = \mathbf{0}$ in the cases $M =$

$(a)\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$; $(b)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$; $(c)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; $(d)\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{pmatrix}$; $(e)\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$.

In cases *(a)* and *(c)*, M has determinant $-1-4 = -5$ and $\cos^2\theta + \sin^2\theta = 1$ respectively, so M is non-singular, and the only solution is the trivial $\boldsymbol{x} = \mathbf{0}$.

In case *(d)*, there is an extra equation, but the first two are the same as in case *(a)*, so there is still only the trivial solution. Note that homogeneous equations can never be inconsistent—there is always the 'trivial' or zero solution.

In case *(b)*, the determinant is $4-4 = 0$, so M is singular and has no inverse [even though it is a square matrix]. Effectively, the equations are $x+2y = 0$ twice; we could, for example, choose y arbitrarily, and then both equations are satisfied by choosing $x = -2y$; this will be a non-trivial solution provided that $y \neq 0$ [as, for example, $x = -2$, $y = 1$].

If there are only two or three equations, you can do this 'by hand'; if you start with many equations, then you will [should!] be using a Gauss–Jordan *tableau*, and then you will find that when you have eliminated as many variables as you can, the back-substitution phase will give you the general solution in terms of the variables for which you have no equation.

For example, in case *(e)*, we have the equations $x+2y+z = 0$ and $2x-y+z = 0$. Eliminating x [subtract twice the first equation from the second] gives $-5y-z = 0$, but we have no further equation to eliminate y. So z [or y, it doesn't matter] can be chosen arbitrarily, then $-5y = z$ or $y = -\frac{1}{5}z$, and $x = -2y-z = \frac{2}{5}z-z = -\frac{3}{5}z$. That is a perfectly good solution, but we can eliminate the fractions by taking $z = 5p$, where p is arbitrary, and then $x = -3p$, $y = -p$, $z = 5p$.

- *Example*: For what value of t does the system of equations

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 2 & -1 & t \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have non-trivial solutions? Find the general solution for this value of t.

Subtracting twice the first row from the third, we have

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 2 & -1 & t \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & -3 & t+2 \end{vmatrix} = 2\times(t+2)-3\times(-3) = 2t+13,$$

so the determinant is zero when $t = -6\frac{1}{2}$.

For this value of t, the Gauss–Jordan *tableau* will produce $0x+0y+0z = 0$ when we eliminate x and y from the equations [check!], so the value of z is arbitrary. In terms of z, the second equation is $2y+3z = 0$, or $y = -\frac{3}{2}z$, and then the first equation is $x+y-z = 0$ or $x = z-y = \frac{5}{2}z$. In other words,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2}z \\ -\frac{3}{2}z \\ z \end{pmatrix} = \tfrac{1}{2}z\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}.$$

As z was arbitrary, so is $\frac{1}{2}z$, so for $t = -6\frac{1}{2}$ the general solution is $x = 5p$, $y = -3p$, $z = 2p$, where p is an arbitrary number. [Check that this solution satisfies all three equations!]

It would be just as good a solution to say $x = -5p$, $y = 3p$, $z = -2p$, or $x = 25p$, $y = -15p$, $z = 10p$, or just $x = \frac{5}{2}z$, $y = -\frac{3}{2}z$; they are all equivalent.

Suppose we generalise a little bit. We have looked at the equations $\boldsymbol{Mx} = \mathbf{0}$ and $\boldsymbol{Mx} = \boldsymbol{b}$. Suppose $\boldsymbol{x}$ occurs also on the right hand side? For example, consider

$$\boldsymbol{Mx} = 3\boldsymbol{x} + \boldsymbol{b},$$

where

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}; \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}; \quad \text{and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Are there any new principles? Not really, because we can take the $\boldsymbol{x}$'s over to the left: instead of

$$\begin{aligned} 1x - 1y &= 3x + 2, \\ -1x + 2y &= 3y + 1, \end{aligned}$$

we have

$$\begin{aligned} (1-3)x - 1y &= 2, \\ -1x + (2-3)y &= 1, \end{aligned}$$

or $N\boldsymbol{x} = \boldsymbol{b}$, where

$$N = \begin{pmatrix} 1-3 & -1 \\ -1 & 2-3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} = M - 3I,$$

where I is the 2×2 identity matrix. So

$$\boldsymbol{x} = N^{-1}\boldsymbol{b} = \ldots = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

or, in other words, $x = -1$ and $y = 0$.

So we can in general solve

$$\boldsymbol{M}\boldsymbol{x} = \lambda\boldsymbol{x}+\boldsymbol{b},$$

where λ is some constant, by solving instead

$$(M-\lambda I)\boldsymbol{x} = \boldsymbol{b}.$$

[Not, however,

$$(M-\lambda)\boldsymbol{x} = \boldsymbol{b};$$

we have not defined what it might mean to subtract a number from a matrix, only how to subtract one matrix from another and how to multiply a matrix by a scalar.]

So in the case where $M-\lambda I$ is non-singular and therefore has an inverse, and where we know λ, there is just one solution,

$$\boldsymbol{x} = (M-\lambda I)^{-1}\boldsymbol{b};$$

and, in the homogeneous case where $\boldsymbol{b} = \mathbf{0}$, there is just the trivial solution $\boldsymbol{x} = \mathbf{0}$. If, on the other hand, $M-\lambda I$ is singular, then we are back to the cases studied earlier.

So we are motivated to look at the homogeneous case where λ is *not* known in advance. This is the equation

$$M\boldsymbol{x} = \lambda\boldsymbol{x}.$$

For any value of λ, there is always the trivial solution $\boldsymbol{x} = \mathbf{0}$. But for some values of λ, there are other solutions, with $\boldsymbol{x} \neq \mathbf{0}$. These values of λ are called *eigenvalues* of M, and the corresponding $\boldsymbol{x}$ are *eigenvectors* of M.