Eigenvectors and eigenvalues

Reminder: If $M\boldsymbol{x} = \lambda\boldsymbol{x}$ has a non-trivial solution with $\boldsymbol{x} \neq 0$, then $\boldsymbol{x}$ is an eigenvector of M with eigenvalue λ. So, we are looking for λ such that the homogeneous equation

$$(M - \lambda I)\boldsymbol{x} = \mathbf{0}$$

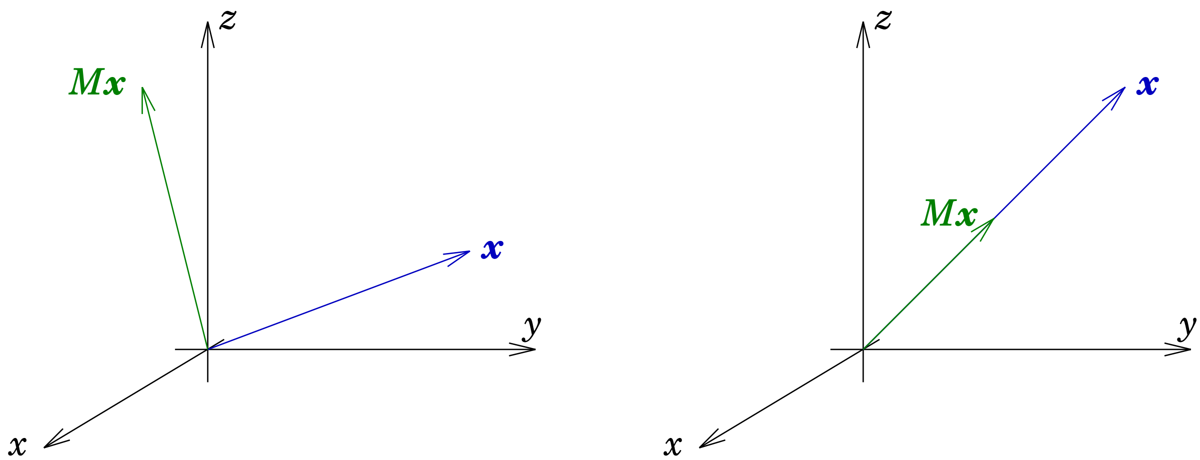
has non-trivial solutions, and this will happen when the determinant of $M - \lambda I$ is zero. This gives a polynomial equation for λ, called the *characteristic* equation of M, whose roots are the eigenvalues. Then, for each eigenvalue, there is a corresponding homogeneous matrix equation which we can solve, as previously, to find the eigenvectors.

See the course booklet for a worked example; more later.

[There are some checks you can make on your working: *(a)* if you add up all the eigenvalues, their sum should be the same as the sum of the diagonal elements of the matrix—the *trace* of the matrix; *(b)* the product of the eigenvalues is the determinant of the matrix; *(c)* you can multiply an eigenvector by the matrix, and you should get the same vector back again, scaled by the eigenvalue. Verify these for the example in the course booklet!]

Warning: If you find yourself doing this in Real Life, then working out the characteristic equation for a large matrix [anything bigger than 3×3], is very hard, except in a few specially simple cases, and this is numerically a Very Bad way to find eigenvalues and eigenvectors. This is a highly-specialised area of Numerical Analysis, and you need professional help and access to a good computer package.

Why is this concept so important/interesting?



(a) $\boldsymbol{x}$ is a 'random' vector (b) $\boldsymbol{x}$ is an eigenvector

The diagram gives us a hint. In general, $\boldsymbol{Mx}$ is a vector that bears no obvious relationship to $\boldsymbol{x}$, especially if $\boldsymbol{x}$ has many components—perhaps hundreds, thousands or worse in practice, rather than the three in the picture. But when $\boldsymbol{x}$ is an eigenvector, $\boldsymbol{Mx}$ is just a scaling of $\boldsymbol{x}$. This means that matrix multiplication is replaced by scalar multiplication, which is much easier to understand.

The central concept of replacing operations you don't understand by scaling applies in many other parts of mathematics. For example, because

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\lambda x} = \lambda \mathrm{e}^{\lambda x},$$

$\mathrm{e}^{\lambda x}$ is an eigen*function* of differentiation with eigenvalue λ, which enables us to replace differentiation by multiplication, and turns differential equations into algebraic ones. This is why we often use exponential functions as 'trial' solutions in differential equations.

For example, suppose we are dealing with small departures from equilibrium of some complicated mechanical structure [e.g. those caused by wind forces on a building]. Because the components of the structure are interacting with each other, the net force on each component depends on the movement of the other components, and we get a ‘Newton’s Second Law’ equation of motion something like

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{M}\boldsymbol{x}$$

[which we don’t easily understand]. If $\boldsymbol{x}$ is an eigenvector, this becomes

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \lambda\boldsymbol{x}$$

[where λ is just a number], which will have exponentially-growing solutions if λ is positive and sine/cosine solutions if λ is negative; these are called *normal mode* solutions of the equation. A general solution of the original equation can usually be written as a combination of normal modes, so the normal modes tell us everything we need to know. Exponentially-growing departures from equilibrium are Bad News if you are designing a building!