Extended example

Consider the matrix

$$M = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

[We'll see an application for this particular matrix later.] Its characteristic equation is given by $\det(M-\lambda I) = 0$, or

$$\begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = 0.$$

Expanding by the top row, this gives

$$(-2-\lambda)\times\Big((-2-\lambda)^2-1\Big)-(-2-\lambda) = 0,$$

or

$$-(2+\lambda)\times\Big((2+\lambda)^2-2\Big) = 0.$$

So, either $\lambda = -2$, or else $(2+\lambda)^2 = 2$, that is, $2+\lambda = \pm\sqrt{2}$. We have three eigenvalues: -2; $-2-\sqrt{2} \approx -3.414$; and $-2+\sqrt{2} \approx -0.586$.

To find the eigenvectors, we look at each eigenvalue in turn.

(a) $\lambda = -2$: In this case, the eigenvector equations are $\boldsymbol{Mx} = -2\boldsymbol{x}$, or

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

in other words,

$$\begin{aligned} -2x + y + 0z &= -2x, \\ x - 2y + z &= -2y, \\ 0x + y - 2z &= -2z, \end{aligned}$$

or $y = 0$, $x = -z$. So an eigenvector is $(1, 0, -1)^T$ [or any non-zero multiple of this].

(b) $\lambda = -2 - \sqrt{2}$: In this case, the eigenvector equations, $\boldsymbol{Mx} = -(2+\sqrt{2})\boldsymbol{x}$, are

$$\begin{aligned} -2x + y + 0z &= -(2+\sqrt{2})x, \\ x - 2y + z &= -(2+\sqrt{2})y, \\ 0x + y - 2z &= -(2+\sqrt{2})z, \end{aligned}$$

or $y = -\sqrt{2}x$, $x + z = -\sqrt{2}y$ and $y = -\sqrt{2}z$ [note that the middle equation is 'the same as' the sum of the other two, apart from a factor of $\sqrt{2}$]. So an eigenvector is $(1, -\sqrt{2}, 1)^T$ [or any non-zero multiple of this].

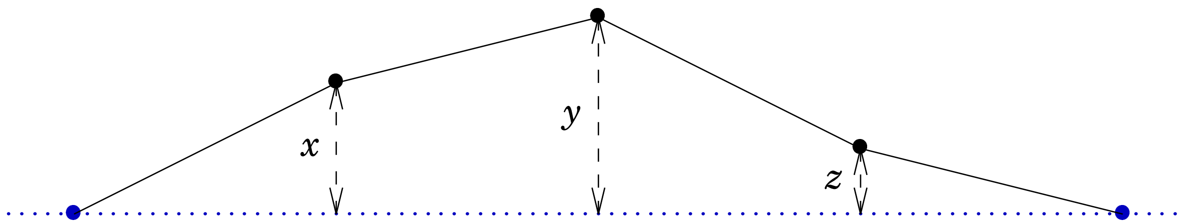
(c) $\lambda = -2 + \sqrt{2}$: Similarly, an eigenvector is $(1, \sqrt{2}, 1)^T$ [details left as an exercise].

Check that these eigenvalues and eigenvectors satisfy the properties 1–4 given in the course booklet. [For property 5, note that, for example, if $B = A^2$ and $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $B\boldsymbol{x} = AA\boldsymbol{x} = A\lambda\boldsymbol{x} = \lambda A\boldsymbol{x} = \lambda^2\boldsymbol{x}$, so $\boldsymbol{x}$ is also an eigenvector of B, but with eigenvalue λ^2.]

OK, so what is the 'application' mentioned earlier?

None of the following *detail* is important or examinable, but you need to know the general idea.

Consider a stretched string with fixed endpoints and with three equal masses equally spaced along it. Imagine that the string is 'plucked' so that the masses are displaced transversely by amounts x, y and z:



How do these masses move? Intuitively, we know that the string will 'twang'; the masses will oscillate to and fro. Left as an exercise in mechanics: the force on the left-hand mass is proportional to $y-2x$, on the central mass is proportional to $x+z-2y$, and on the right-hand mass is proportional to $y-2z$. In other words,

$$\ddot{x} = k(y-2x), \;\; \ddot{y} = k(x+z-2y), \;\; \text{and } \ddot{z} = k(y-2x),$$

where the dots indicate time derivatives, and k is some constant [related to the masses and the string tension]. In other words,

$$\ddot{\boldsymbol{x}} = kM\boldsymbol{x},$$

where M is the matrix above and $\boldsymbol{x} = (x, y, z)^T$. [Note that $\boldsymbol{x}$ is not 'the position' of any one of the masses, but is a vector whose components are the displacements of all of them.]

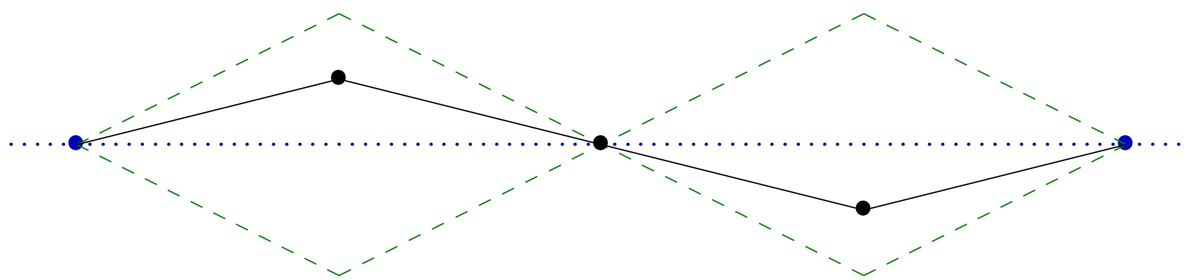
Now this differential equation is quite messy to solve. But if $\boldsymbol{x}$ is an eigenvector of M with eigenvalue λ, then it becomes

$$\ddot{\boldsymbol{x}} = k\lambda\boldsymbol{x},$$

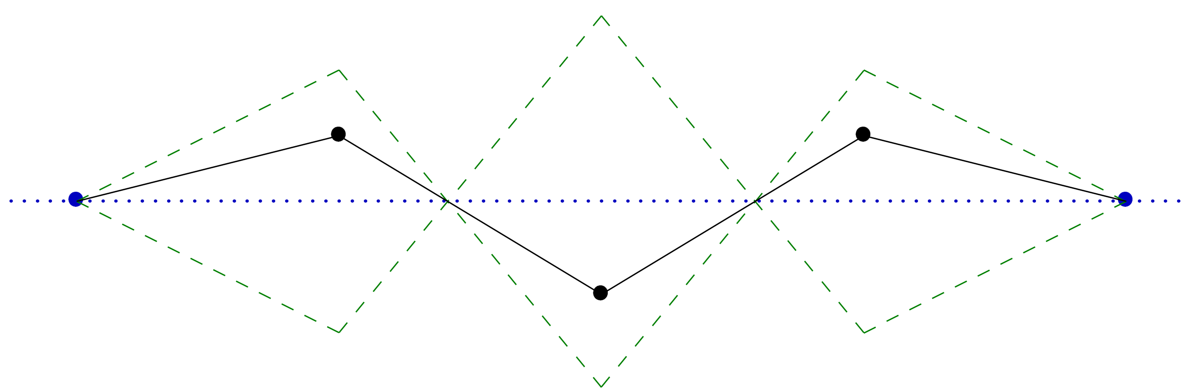
which is the differential equation for SHM, 'simple harmonic motion'; the matrix M has disappeared, and we can solve the equation.

What do the solutions look like? Basically, like the eigenvectors, but 'oscillating' in size with frequency $\sqrt{-k\lambda}$.

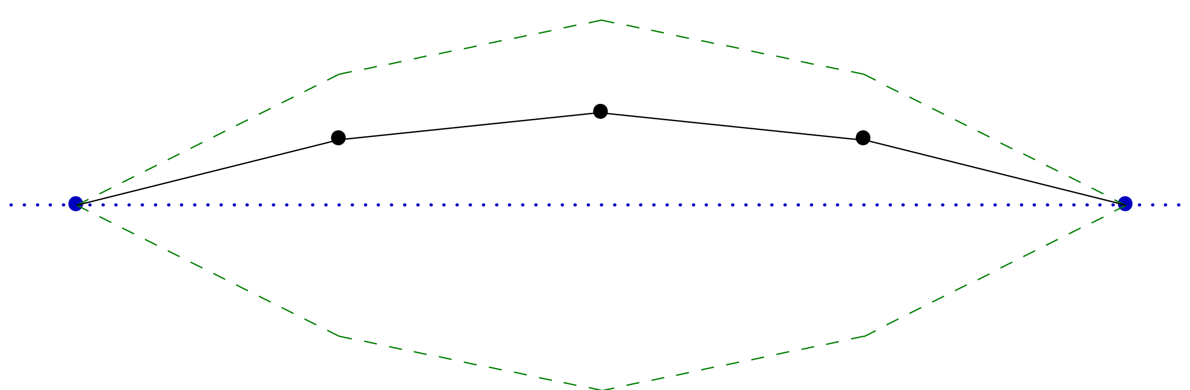
(a) Corresponding to the eigenvector $(1, 0, -1)^T$, we have:



(b) Corresponding to the eigenvector $(1, -\sqrt{2}, 1)^T$, we have:



(c) And corresponding to the eigenvector $(1, \sqrt{2}, 1)^T$, we have:



Any combination whatsoever of positions of the masses can be represented as a weighted sum of these eigenvectors, so that any motion can be represented as a combination of the motions described above. Effectively, we can decompose any solution of this problem in mechanics into these three more basic solution, which proceed independently [and orthogonally—check!].

It is very usual that a complicated mechanics problem can be reduced to a collection of more basic problems, and the general solution expressed as a sum of solutions of these. It is also very usual that the eigenvectors seem to relate to these basic solutions and to interesting features of the physics/engineering as well as to 'mere' mathematics.

For example, in the present problem, in real life there will be 'friction' damping the motion. This will dampen the high-frequency motions faster than the low-frequency motions, so after a while, only the 'fundamental' [solution (c) above] will remain. [You can make a skipping rope, eg, oscillate rather like solutions (a) or (b), but it's much harder than (c).]

If we extend the problem to four, five, ..., a hundred, ... masses equally spaced on a light string/spring, then in the limit we get something very like a uniform heavy spring. The corresponding eigenvectors then look more and more like the functions $\sin x$ [the 'fundamental'], $\sin 2x$ [the 'first harmonic'], $\sin 3x$, $\sin 4x$ and so on.

In music, the proportions of the various harmonics and how they decay through 'friction' determines what a note sounds like; the proportions are obtained by Fourier analysis [beyond the scope of this module] of the initial string shape when it is plucked, bowed or hit. But the same basic equations occur not only with sound waves, both transverse and longitudinal, but also with water waves and with electromagnetic waves; and with, for example, the equations governing the oscillations of a building in a high wind [where the fundamental period of oscillation may be many seconds].

It was obtaining the 'wave equations' from his theory of electricity, with waves travelling at the speed of light, that led Maxwell to infer the electromagnetic nature of light and the spectrum, and that led to the discovery of radio waves. It's very common in maths that finding the same equations in different contexts leads to physical inference about the nature of phenomena.