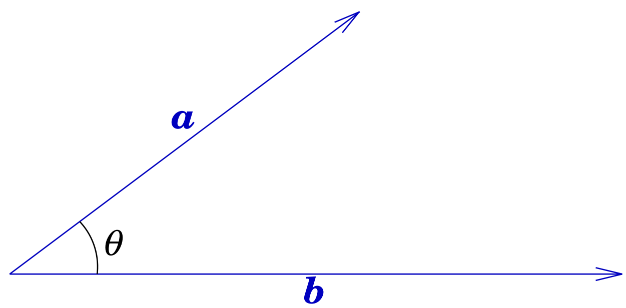
Scalar products

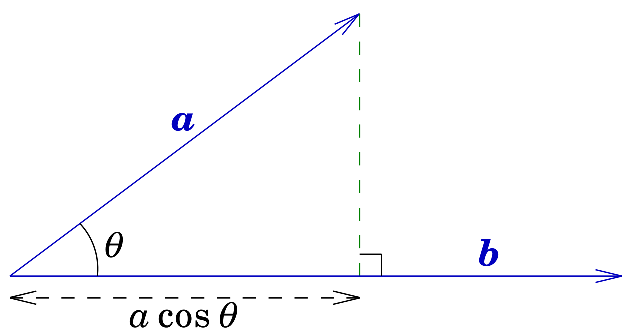
Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, their *scalar*, or *inner*, or *dot*, product is

$$\boldsymbol{a}\cdot\boldsymbol{b} \;=\; a\,b\cos\theta,$$

where a is the size of $\boldsymbol{a}$, b is the size of $\boldsymbol{b}$ and θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



It's just a number, not a vector. It has many applications in practice as $a\cos\theta$ is the *component* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:



[and equally $b\cos\theta$ is the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$]. So $\boldsymbol{a}\cdot\boldsymbol{b}$ is the size of $\boldsymbol{b}$ times the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$, especially useful if $\boldsymbol{b}$ is a unit vector.

Properties:

- $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\theta = 0$ and $\boldsymbol{a}\cdot\boldsymbol{b} = a\,b$.
- [Special case:] If $\boldsymbol{b} = \boldsymbol{a}$, then $\boldsymbol{a}\cdot\boldsymbol{a} = a^2$ is the square of the size of $\boldsymbol{a}$. This is a good way to find sizes of vectors.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\theta = \pi/2$ and $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. This is a good way to test for vectors being at right angles.
- The unit vectors $\boldsymbol{i} = (1,0,0)$, $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$ have size 1 and are perpendicular to each other. So

$$\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{i} = \boldsymbol{k}\cdot\boldsymbol{j} = \boldsymbol{i}\cdot\boldsymbol{k} = 0,$$

$$\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1.$$

Example

$$\begin{aligned}&(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})\cdot(3\boldsymbol{i}+4\boldsymbol{k})\\ &\quad= 6\boldsymbol{i}\cdot\boldsymbol{i}+3\boldsymbol{j}\cdot\boldsymbol{i}-6\boldsymbol{k}\cdot\boldsymbol{i}+8\boldsymbol{i}\cdot\boldsymbol{k}+4\boldsymbol{j}\cdot\boldsymbol{k}-8\boldsymbol{k}\cdot\boldsymbol{k}\\ &\quad= 6+0-0+0+0-8 = -2.\end{aligned}$$

[With practice, you may be able to leave out the intermediate steps.]

More generally,

$$(l,m,n)\cdot(p,q,r) = lp+mq+nr.$$

Advice: Use co-ordinates [list notation] if that's what you're given or if you have to, but don't rush. You often know things about sizes and directions of [eg] forces and velocities that enable you to use the definition directly.

Examples

If

$$\begin{aligned} \boldsymbol{a} &= 2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k} \text{ and} \\ \boldsymbol{b} &= 3\boldsymbol{i}+4\boldsymbol{k}, \end{aligned}$$

[or, if you prefer, $\boldsymbol{a} = (2,1,-2)$, $\boldsymbol{b} = (3,0,4)$], then:

- What is the size of $\boldsymbol{a}$?

  We have $a^2 = \boldsymbol{a}\cdot\boldsymbol{a} = 2^2+1^2+(-2)^2 = 4+1+4 = 9$, so the size of $\boldsymbol{a}$ is $a = \sqrt{9} = 3$.

  Similarly, the size of $\boldsymbol{b}$ is $\sqrt{3^2+4^2} = 5$ and, in general, the size of (l,m,n) is $\sqrt{l^2+m^2+n^2}$, as before.

- What is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$?

  From a previous example, $\boldsymbol{a}\cdot\boldsymbol{b} = -2$. This is also $a\,b\cos\theta = 3\times5\times\cos\theta$, so $\cos\theta = -2/15$ and [calculator] $\theta = 1.7045\ldots$ radians or nearly 98 degrees.

- What is the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$?

  The component is $a\cos\theta = \boldsymbol{a}\cdot\boldsymbol{b}/b = -2/5$.

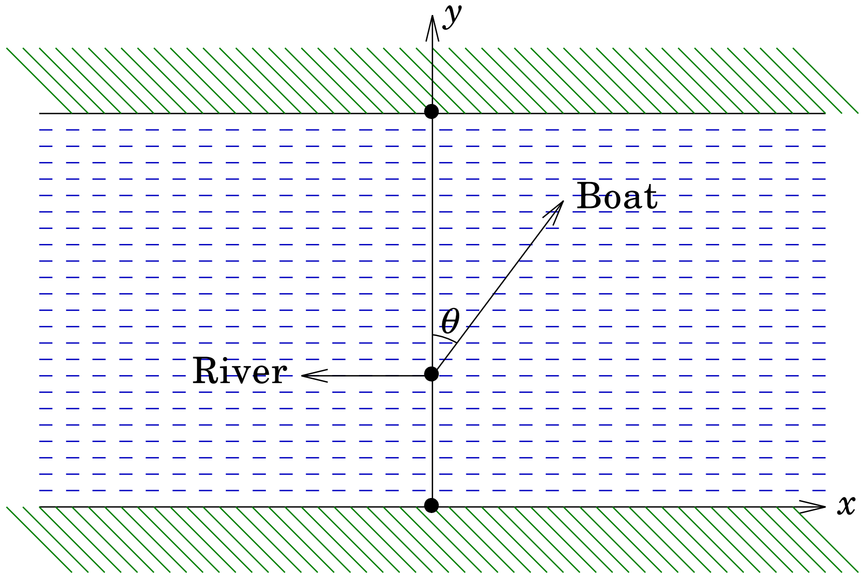
  Note that the components of $\boldsymbol{a}$ in the x-, y- and z-directions are $\boldsymbol{a}\cdot\boldsymbol{i} = 2$, $\boldsymbol{a}\cdot\boldsymbol{j} = 1$ and $\boldsymbol{a}\cdot\boldsymbol{k} = -2$ respectively, giving us the relationship between vectors as components or lists and vectors as size and direction.

  In particular, if R is the point at position (x,y,z), then its position vector is

$$\overrightarrow{OR} = \boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}.$$

Example

A ferry boat can travel at 5 km/h, and is trying to cross a river which is flowing at 3 km/h. At what angle θ should the boat head upstream in order to cross the river from one bank to the point directly opposite on the other bank?



The velocity $\boldsymbol{b}$ of the boat has to satisfy that $\boldsymbol{b} - 3\boldsymbol{i}$ is perpendicular to the river bank, *i.e.* to $\boldsymbol{i}$; so $(\boldsymbol{b} - 3\boldsymbol{i})\cdot\boldsymbol{i} = 0$, or $\boldsymbol{b}\cdot\boldsymbol{i} = 3$. So

$$5\cos(\pi/2 - \theta) = 5\sin\theta = 3,$$

so that $\theta = \sin^{-1} 0.6 = 0.64\ldots$ radians or 37 degrees.

Not asked for, but the component of the velocity of the boat across the stream is $\boldsymbol{b}\cdot\boldsymbol{j} = 5\cos\theta = 4$ km/h.
Note that if the river is flowing faster than the boat, then the boat cannot hold its line across the river—but real rivers flow slower near the banks, which may allow the boat to crawl upstream once it has [nearly] crossed. Further note: there are several ways of seeing these results!