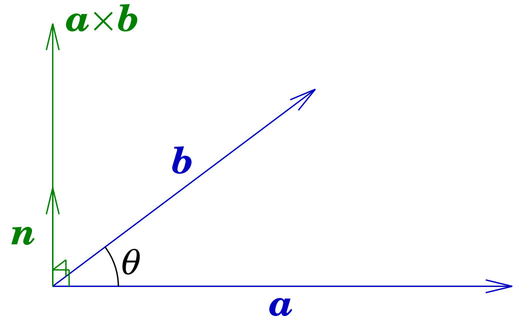
Vector products

Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, their *vector*, or *cross*, product is

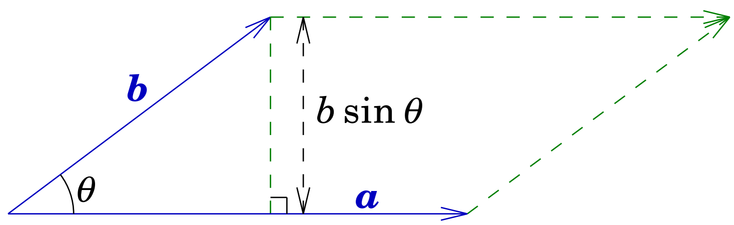
$$\boldsymbol{a}\times\boldsymbol{b} = a\,b\sin\theta\,\boldsymbol{n},$$

where a is the size of $\boldsymbol{a}$, b is the size of $\boldsymbol{b}$ and θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{n}$ is the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.



Wait! *The* vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$? There are *two* such vectors, one 'up', one 'down'. We choose the one such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{n}$ form a right-handed triple of vectors. As this is an arbitrary decision that the real world doesn't have to agree with, physical laws cannot depend on it.

It has many applications in practice as $b\sin\theta$ is the perpendicular distance of the [end of] $\boldsymbol{b}$ from [the line of action of] $\boldsymbol{a}$:



[and $a\sin\theta$ is the distance of $\boldsymbol{a}$ from $\boldsymbol{b}$]. So the size of $\boldsymbol{a}\times\boldsymbol{b}$ is the area of the parallelogram based on $\boldsymbol{a}$ and $\boldsymbol{b}$ ['base times perpendicular height']; and $\boldsymbol{a}\times\boldsymbol{b}$ itself is a [pseudo-]vector representing that area. [Remember that areas are 'almost' vectors?]

Properties:

- $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$. [The right-hand rule applies the other way.] This result shows that we need to be **very careful** when doing algebra with vector products—it is *essential* to keep factors in the right order.

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\theta = 0$ and $\boldsymbol{a}\times\boldsymbol{b} = \mathbf{0}$. This is a good way to test for vectors being parallel.

- The unit vectors $\boldsymbol{i} = (1,0,0)$, $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$ have size 1, are perpendicular to each other, and in that order form a right-handed triple. So

$$\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \mathbf{0};$$
$$\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j},$$
$$\boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k},\ \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i},\ \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}.$$

Example

$$\begin{aligned}
&(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})\times(3\boldsymbol{i}+4\boldsymbol{k})\\
&\quad= 6\boldsymbol{i}\times\boldsymbol{i}+3\boldsymbol{j}\times\boldsymbol{i}-6\boldsymbol{k}\times\boldsymbol{i}+8\boldsymbol{i}\times\boldsymbol{k}+4\boldsymbol{j}\times\boldsymbol{k}-8\boldsymbol{k}\times\boldsymbol{k}\\
&\quad= 6\mathbf{0}-3\boldsymbol{k}-6\boldsymbol{j}-8\boldsymbol{j}+4\boldsymbol{i}-8\mathbf{0}\\
&\quad= 4\boldsymbol{i}-14\boldsymbol{j}-3\boldsymbol{k}.
\end{aligned}$$

[With practice, you may be able to leave out the intermediate steps.]

It's always worth checking that the result is perpendicular to the vectors you started with:

$$(4,-14,-3)\cdot(2,1,-2) = 8-14+6 = 0,$$
$$(4,-14,-3)\cdot(3,0,4) = 12-12 = 0;$$

it's not fool-proof, but it's a quick and easy test, and will show up most sign errors and arithmetic blunders.

More generally,

$$(l, m, n)\times(p, q, r) = (mr - nq)\boldsymbol{i} + (np - lr)\boldsymbol{j} + (lq - mp)\boldsymbol{k}.$$

That's pretty horrible, so almost no-one remembers it that way. Instead, notice the cross-products, and think of determinants:

$$(l, m, n)\times(p, q, r) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ l & m & n \\ p & q & r \end{vmatrix}.$$

You should check that this is the same result, and also check the previous numerical example; be particularly careful about signs, especially when working out the coefficient of $\boldsymbol{j}$. [Some people and books treat this expression as the *definition* of the vector/cross product.]

Note that swapping the order of $\boldsymbol{a}$ and $\boldsymbol{b}$ corresponds to swapping two rows of the determinant, and so changes the sign; also that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then the second and third rows are proportional to each other, and we can subtract a multiple of one row from the other to give a row of zeros, so that the result is $\mathbf{0}$, as expected.

Advice: Use co-ordinates [list notation] if that's what you're given or if you have to, but don't rush. You often know things about sizes and directions of [eg] forces and velocities that enable you to use the definition directly.

Examples

- $$(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b})$$
  $$= \boldsymbol{a}\times\boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{a} - \boldsymbol{a}\times\boldsymbol{b} - \boldsymbol{b}\times\boldsymbol{b}$$
  $$= \mathbf{0} - \boldsymbol{a}\times\boldsymbol{b} - \boldsymbol{a}\times\boldsymbol{b} - \mathbf{0} = -2\boldsymbol{a}\times\boldsymbol{b}.$$

  Note that this is quite different algebraically from

  $$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})$$
  $$= \boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{a}-\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{b}\cdot\boldsymbol{b} = a^2-b^2,$$

  very similar to ordinary 'difference of squares' algebra.

- If a force $\boldsymbol{F}$ acts at a point $\boldsymbol{r}$, then its moment [turning effect] about the origin is $\boldsymbol{r}\times\boldsymbol{F}$. [The size of this is therefore the size of the force times the distance of its line of action from the origin.] For a body to be in equilibrium, not only must the forces on it add to $\mathbf{0}$, but so also must their moments—this is the three-dimensional version of the balance ['see-saw'] law. [Compare also the 'ladder against the wall' problem.]

- What is the area of the triangle whose vertices are the origin and the points $(2,1,-2)$ and $(3,0,4)$?

  As previously, $(2,1,-2)\times(3,0,4) = (4,-14,-3)$. This is the vector representing the area of the parallelogram based on the vectors $(2,1,-2)$ and $(3,0,4)$. So the size of that area is $\sqrt{4^2+(-14)^2+(-3)^2} = \sqrt{16+196+9} = \sqrt{221} \approx 14.86$. The triangle is half of the parallelogram, so its area is about 7.43.

  Exercise: We already know the angle between these vectors and their lengths, from previous examples; so check this answer using ordinary trigonometry.

Notes:

- If we look at two-dimensional problems, *e.g.* with all the 'action' taking place in the xy-plane, then all the 'interesting' vectors—the forces, velocities, and so on—are also in that plane, and any cross-products are perpendicular to that plane [so in the z-direction]. For example,

$$(2\boldsymbol{i}-\boldsymbol{j})\times(\boldsymbol{i}+2\boldsymbol{j}) \;=\; 4\boldsymbol{i}\times\boldsymbol{j}-\boldsymbol{j}\times\boldsymbol{i} \;=\; 5\boldsymbol{k}.$$

- You may be thinking that the dot and cross products are a bit arbitrary, and that there ought to be many other ways of combining sizes and directions. But it turns out that *(a)* there are philosophical and physical reasons why most other ways are of no practical use, and *(b)* dot and cross products occur in many physical laws. Recognising them as part of the 'modelling' process when thinking about real-world problems, and then understanding their properties, makes life much easier. But *(c)* there is a third way, the *tensor* or *outer* product of two vectors; not explored here, but sometimes useful in tensor algebra/calculus.