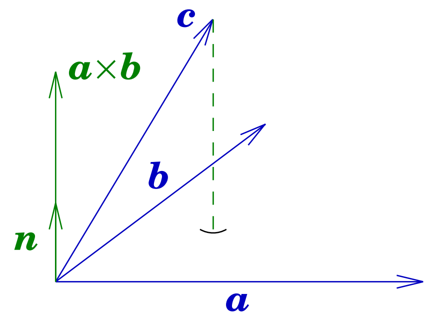
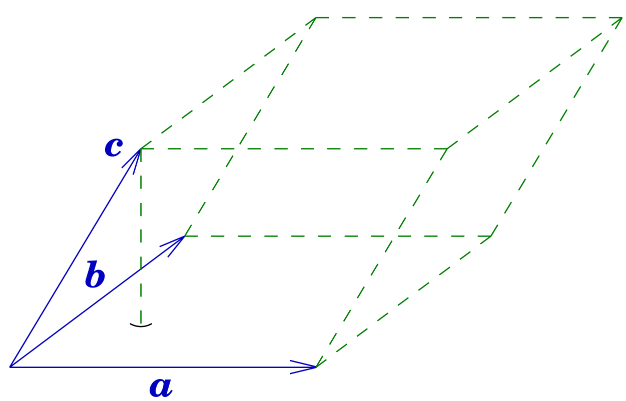
Scalar triple products

Given three vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, we can take the vector product of any two, say $\boldsymbol{a}\times\boldsymbol{b}$, and take either the scalar or vector product with the third. Here we consider only the scalar product, *e.g.* $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.



Note that the component of $\boldsymbol{c}$ in the direction of $\boldsymbol{n}$ is the height of $\boldsymbol{c}$ above the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$, so $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ is the *volume* of the *parallelepiped* [skew block] based on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ['area of base times height']:



- We can use this result to find volumes.

  It is also a good way to find the distance of a point from a plane: the distance of $\boldsymbol{c}$ from the plane formed by $\boldsymbol{a}$ and $\boldsymbol{b}$ is $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}/|\boldsymbol{a}\times\boldsymbol{b}|$, the volume divided by the area of the base.

  Special case: the distance is zero, that is, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar, if $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = 0$.

- There are twelve different ways of ordering the vectors and the products; these all give either the same volume or its negative, depending on whether $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are right- or left-handed. $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ is often written $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$.

- Note that

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

  where a_1 is the x-component of $\boldsymbol{a}$, and so on. *Exercise*: show this! This is often a convenient way to calculate scalar triple products; but also quite often you need $\boldsymbol{a}\times\boldsymbol{b}$ anyway.

- *Examples*: Find the volume of the parallelepiped three of whose sides are the vectors $(2,1,-2)$, $(3,0,4)$ and $(1,-1,3)$.

  As previously, $(2,1,-2)\times(3,0,4) = (4,-14,-3)$, so the required volume is $(4,-14,-3)\cdot(1,-1,3) = 4+14-9 = 9$.

  Or, the volume is

$$((2,1,-2)\times(3,0,4))\cdot(1,-1,3) = \begin{vmatrix} 2 & 1 & -2 \\ 3 & 0 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 8-5+6 = 9.$$

- For what value of k are the vectors $(2,1,-2)$, $(3,0,4)$ and $(1,-1,k)$ co-planar?

  The volume of the parallelepiped formed by those three vectors is [as above] $(4,-14,-3)\cdot(1,-1,k) = 4+14-3k = 18-3k$, so the volume is zero, and the vectors co-planar, when $k = 6$.

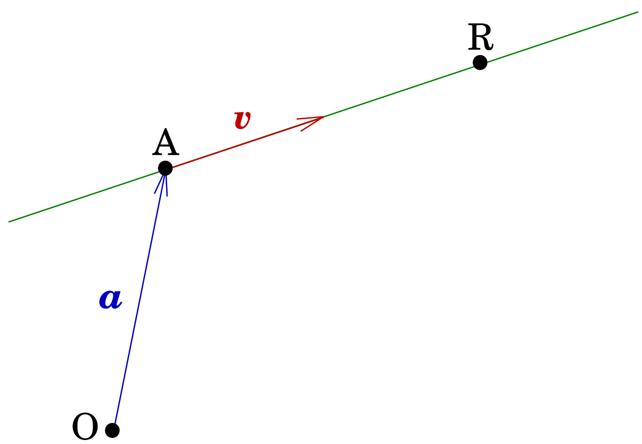
Vector Geometry

We look especially at *lines* and *planes*. Other parts of geometry are concerned with *angles* and *lengths* [for which we can use dot products], and with curves or surfaces such as *circles*, *spheres* and *ellipses* [for which we can use co-ordinate geometry].

Lines

- May be defined by two points ['Find the line between A and B'] or by a point and a direction ['Find the line through A in the direction of the vector $\boldsymbol{v}$'].

- The second is slightly easier:

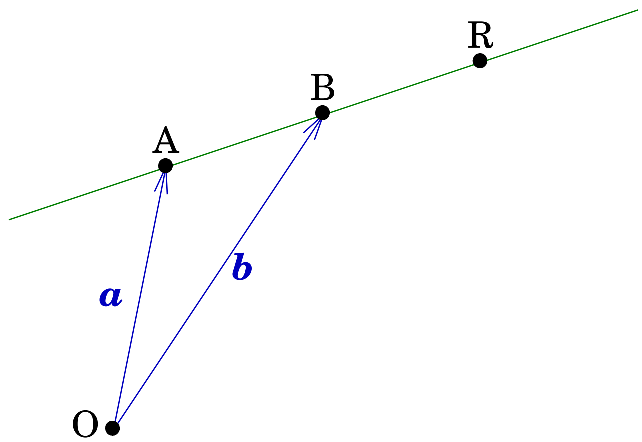


A typical point, R, on the line has a position vector $\boldsymbol{r}$ such that $\boldsymbol{r}-\boldsymbol{a}$ is scaled from $\boldsymbol{v}$; it is in the same direction, but has a different length. So $\boldsymbol{r}-\boldsymbol{a} = t\boldsymbol{v}$, or $\boldsymbol{r} = \boldsymbol{a}+t\boldsymbol{v}$, where t is some scalar. This is the *parametric* equation of the line, with parameter t, $-\infty < t < \infty$.

- If $\boldsymbol{v}$ is a *unit* vector, then its components are the 'direction cosines' of the line; see the sheet on these in the module booklet. For example, $\boldsymbol{v}\cdot\boldsymbol{i} = \cos\alpha$, where α is the angle between the line and the x-axis. If $\boldsymbol{v}$ is *not* a unit vector, then we need to scale it first; compare the second paragraph on the sheet.

- If t represents time, and $\boldsymbol{v}$ velocity, then $\boldsymbol{r}$ as just obtained is the position of a particle that is at A at time $t = 0$ and moving with a constant velocity $\boldsymbol{v}$.

- To find the line between A and B, note that the vector $\boldsymbol{b} - \boldsymbol{a}$ plays the same role as the vector $\boldsymbol{v}$ in the previous version:



So the equation of the line is $\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) = (1-t)\boldsymbol{a} + t\boldsymbol{b}$. This could, for example, represent a particle moving with constant velocity which is at A at time $t = 0$ and at B at time $t = 1$.

- To find the cartesian equation of the line, replace the vectors by their components, and use these to eliminate t [see example below]. Note that in three dimensions, a one-dimensional construction, such as a straight line, needs $3-1 = 2$ equations.

Examples

- What is the parametric form of the equation of the straight line between the points $(2, -2, 1)$ and $(4, 1, -1)$?

  Note: Questions of this sort are often 'dressed up': A boat is at the point (...), it observes that a seagull is in line with a lighthouse at (...) [so the seagull is somewhere along the line] and ....

  So the vector $(4, 1, -1) - (2, -2, 1) = (2, 3, -2)$ is in the direction of the line, and the equation of the line is

  $$\boldsymbol{r} = (2, -2, 1) + t(2, 3, -2) = (2+2t)\boldsymbol{i} + (-2+3t)\boldsymbol{j} + (1-2t)\boldsymbol{k}$$

  [either version acceptable].

- What is the parametric equation of the line through the point $(2, -2, 1)$ in the direction of the vector $2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$?

  [The same!]

- What is the cartesian equation of this line?

  If $\boldsymbol{r} = (x, y, z)$, then we have

  $$x = 2+2t; \;\; y = -2+3t; \text{ and } z = 1-2t.$$

  In other words, solving each equation for t,

  $$\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-1}{-2}.$$

- What are the direction cosines of this line?

  The length of $2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ is $\sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$, so they are $2/\sqrt{17}$, $3/\sqrt{17}$ and $-2/\sqrt{17}$. The inverse cosines of these values would give us the angles the line makes with the x-, y- and z-axes.