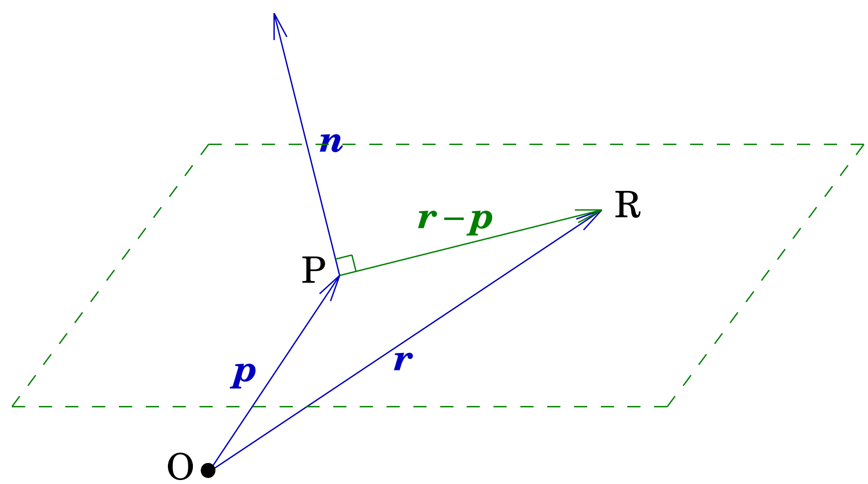
Planes

We can ask for the plane that passes through a given point and is in a given orientation [direction]; or that passes through a given point and contains two vectors [why two?]; or that passes through three points; and we can ask for the answer as a vector equation or in cartesian form.

- Given a point P with position vector $\boldsymbol{p}$, we can specify the direction of a plane through P by giving a vector $\boldsymbol{n}$ for a normal to it. Any normal vector [except $\mathbf{0}$] will do!



In that case, a typical point R in the plane is characterised by the fact that the vector $\overrightarrow{PR} = \boldsymbol{r} - \boldsymbol{p}$ lies in the plane, and so is perpendicular to the normal vector $\boldsymbol{n}$. In other words,

$$(\boldsymbol{r}-\boldsymbol{p})\cdot\boldsymbol{n} \;=\; 0.$$

This is the vector equation of the plane through P and normal [perpendicular] to $\boldsymbol{n}$.

- Given a point P and two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we can find the plane that contains them by constructing $\boldsymbol{n} = \boldsymbol{u}\times\boldsymbol{v}$, which is therefore normal to the plane, and then using the previous result. So the equation of the plane is

$$(\boldsymbol{r}-\boldsymbol{p})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = 0.$$

  Equivalently, this is a scalar triple product:

$$[\boldsymbol{r}-\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{v}] = 0,$$

  specifying that the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{r}-\boldsymbol{p}$ are coplanar.

- Given three points, P, S and T, we can find the plane through them by constructing $\boldsymbol{u} = \overrightarrow{PS}$, $\boldsymbol{v} = \overrightarrow{PT}$, and then using the previous result. So the equation of the plane is

$$[\boldsymbol{r}-\boldsymbol{p}, \boldsymbol{s}-\boldsymbol{p}, \boldsymbol{t}-\boldsymbol{p}] = 0,$$

  where $\boldsymbol{s}$ and $\boldsymbol{t}$ are the position vectors of S and T, specifying that the vectors $\boldsymbol{r}-\boldsymbol{p}$, $\boldsymbol{s}-\boldsymbol{p}$ and $\boldsymbol{t}-\boldsymbol{p}$ are coplanar.

  You will get the same result, despite the different appearance, if you construct, say, $\boldsymbol{u} = \overrightarrow{ST}$, or use $\boldsymbol{r}-\boldsymbol{t}$, or whatever. Verify this using the example below!

- To find the cartesian form of the equation, just replace $\boldsymbol{r}$ by (x, y, z) in the vector form.

Examples

- Find the equation of the plane through the points $P = (1,1,1)$, $S = (2,-1,0)$ and $T = (-1,2,-2)$.

  Two vectors in the plane are $\overrightarrow{PS} = (2,-1,0)-(1,1,1) = (1,-2,-1)$ and $\overrightarrow{PT} = (-1,2,-2)-(1,1,1) = (-2,1,-3)$. So a normal vector to the plane is

$$\overrightarrow{PS}\times\overrightarrow{PT} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & -1 \\ -2 & 1 & -3 \end{vmatrix} = (6+1)\boldsymbol{i}-(-3-2)\boldsymbol{j}+(1-4)\boldsymbol{k} = 7\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k}.$$

  Check: $(7,5,-3)\cdot(1,-2,-1) = 7-10+3 = 0$, and $(7,5,-3)\cdot(-2,1,-3) = -14+5+9 = 0$.

  So the equation of the plane is

$$(\boldsymbol{r}-(1,1,1))\cdot(7,5,-3) = 0.$$

  Since $(1,1,1)\cdot(7,5,-3) = 7+5-3 = 9$, this is equivalently

$$\boldsymbol{r}\cdot(7,5,-3) = 9.$$

  If we want this in cartesian form, replace $\boldsymbol{r}$ by (x,y,z); then $(x,y,z)\cdot(7,5,-3) = 7x+5y-3z$, so the equation of the plane is

$$7x+5y-3z = 9.$$

  Check: This equation is satisfied by the three points we started with.

  Note: We can also go the other way; if we have the equation of a plane in a form such as $7x+5y-3z = 9$, then we know that the vector $7\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k}$ is normal to the plane, and there is a vector form $\boldsymbol{r}\cdot(7\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k}) = 9$.

- How far is the point $R = (2, -1, 3)$ from the plane just found?

  So the vector $\overrightarrow{PR} = (2, -1, 3) - (1, 1, 1) = (1, -2, 2)$. By construction, the vector $7\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$ is normal to the plane; its size is $\sqrt{7^2 + 5^2 + (-3)^2} = \sqrt{83}$. So the component of $\overrightarrow{PR}$ in the direction normal to the plane is

  $$\overrightarrow{PR} \cdot (7\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k})/\sqrt{83} = (7 - 10 - 6)/\sqrt{83} = -9/\sqrt{83} \approx -0.988.$$

  So the distance of R from the plane is the absolute value of the component just found, or approximately 0.988 units. [R is on the opposite side of the plane from the normal.]

- A light ray from R strikes the plane just found at P. What is its angle of incidence?

  We want the angle between the line PR and the normal at P; so between the vectors $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $7\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$. These vectors have lengths 3 and $\sqrt{83}$, and dot product -9, as just found, so the angle between is $\cos^{-1}(-9/(3\sqrt{83})) \approx 109$ degrees. As this is greater than 90, equivalently the ray is striking at $180 - 109 = 71$ degrees on the opposite side to the normal. [From this, we could easily do nice things with the optics if we wanted to know where the ray got reflected or refracted to.]

- What is the angle between the plane just found and the plane $x - 2y + 2z = 1$?

  Normals to the planes are $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $7\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$; as just found, the angle between these is 71 [or 109] degrees.