The University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

A LEVEL 3 MODULE, SPRING 2006–2007

COMPUTERISED MATHEMATICAL METHODS IN ENGINEERING

Time allowed TWO hours DRAFT

Candidates must NOT start writing their answers until told to do so.

This paper contains FIVE questions which carry equal marks.
Full marks may be obtained for FOUR complete answers.
Credit will be given for the best FOUR answers.

An indication is given of the approximate weighting of each section of a question by means of a figure enclosed by square brackets, eg [12], immediately following that section

Only silent, self-contained calculators with a Single-line Display or Dual-line Display are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: *Formula sheet*

1 (*a*) Briefly explain the terms *truncation error*, *rounding error* and *cancellation error*, giving an example of each. [6]

(*b*) Find correct to FIVE places of decimals all real roots of the equations

(i) $x^3 - 3x - 1 = 0;$ [11]

(ii) $x^2 = \sin x.$ [8]

[Make sure your calculator is set to use radians. You may find it helpful to draw graphs to determine the number and approximate locations of the roots. You need not record every step in your calculations, but you should give sufficient detail to demonstrate that you are using an appropriate technique and understand how it works.]

2 (*a*) Briefly explain the conditions under which the composite form of Simpson's Rule may be expected to work well to find a numerical integral, and suggest two ways, with examples, in which integrals for which Simpson's Rule is unsuitable might be transformed to make them suitable. [9]

(*b*) Evaluate

$$\int_0^1 \log \sin x \, \mathrm{d}x$$

correct to FOUR places of decimals. [16]

[HINT: Integrate by parts: what is $\dfrac{\mathrm{d}}{\mathrm{d}x} x \log \sin x$?]

3 (*a*) Use the method of Gaussian elimination to solve the system of equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

where

$$A = \begin{pmatrix} 4 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 4 & 1 \end{pmatrix}; \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}; \quad \text{and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

[12]

From your solution, write down the determinant of A. [3]

Briefly explain how you would adapt your solution to find the inverse of A. [You are not asked to find the inverse.] [3]

(*b*) Explain what is meant by *partial pivoting* and why it is important in Gaussian elimination. Illustrate your explanation by reference to the solution of

$$\begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

in the case where ε is very small. [7]

4 (*a*) Briefly describe how the Adams–Bashforth three-point formula,

$$y_1 \approx y_0 + \tfrac{1}{12}h(23f_0 - 16f_{-1} + 5f_{-2}),$$

[with standard notation] may be used in conjunction with the Adams–Moulton formula

$$y_1 \approx y_0 + \tfrac{1}{24}h(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

in the numerical solution of the differential equation $\dfrac{dy}{dx} = f(x,y)$. [6]

What are the advantages and disadvantages of this method compared with the Runge–Kutta method? [4]

(*b*) Use the Adams–Bashforth/Adams–Moulton method to estimate the value of y when $x = 1$ given that $\dfrac{dy}{dx} = xy$ and that $y = 1$ when $x = 0$, using a step-length $h = 0.2$. You may assume that the values $y(\pm 0.2) = e^{0.02}$ have been found to sufficient accuracy by some other method. [15]

5 It is desired to solve the wave equation,

$$\frac{\partial^2\phi}{\partial t^2} = c^2 \frac{\partial^2\phi}{\partial x^2}$$

for the function $\phi(x,t)$, given initial values of ϕ and $\dfrac{\partial\phi}{\partial t}$ at time $t = 0$.

(*a*) Show that an approximate solution to the wave equation may be derived from the system of equations

$$u_{i,j+1} \approx 2u_{i,j} - u_{i,j-1} + (kc/h)^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}),$$

where $u_{i,j} \approx \phi(hi,kj)$ for suitable strip-widths h and k, and where $j \geqslant 1$. [7]

[You may assume the approximation $\dfrac{d^2f}{dx^2} \approx (f(x-h) - 2f(x) + f(x+h))/h^2$.]

(*b*) Explain briefly why the above system of equations cannot be used when $j = 0$, and what equations should be used instead. [6]

(*c*) A stretched string is displaced and released from rest; its displacement ϕ satisfies the wave equation, the initial displacement $\phi(x,0) = \cos x$ in units such that $c = 1$, and the string is held fixed at $x = \pm\pi/2$. Write down the above equations for $u_{i,j}$ in the case $h = \pi/4$, $k = 1$, distinguishing the cases $j = 0$ and $j \geqslant 1$. Find $\phi(x,3)$ in this approximation, for $-\pi/2 \leqslant x \leqslant \pi/2$. [10]

Comment on the stability of these equations given these choices for h and k. [2]