HGCMCE—Computerised Mathematical Methods in Engineering

Themes

- **Processes for getting numbers out of problems**

  — pencil and paper, calculator, computer program, package (such as Maple)

- **Errors**

  — knowledge, control. ‘42’.

- **Efficiency**

  — speed, simplicity, robustness

- **Healthy scepticism**

  — ‘the computer says’

- **Limitations**

  — when you need an expert

- **Numbers vs analysis**

  — when to switch from calculus/algebra to numerical work

Example 1

(a) $\dfrac{dy}{dx} = y \sin x$

$$y = A e^{-\cos x}$$

[useful]

(b) $\dfrac{dy}{dx} = y \sin x + x$

$$y = e^{-\cos x} \int x \, e^{\cos x} dx$$

[possibly slightly useful]

(c) $\dfrac{dy}{dx} = y^2 \sin x + x$

$$y = ??$$

[useless]

In numerical work, to solve $\dfrac{dy}{dx} = f(x, y)$, it matters little what f is, within reason. All of the above can be solved numerically to comparable accuracy in comparable time. Equation (c) is *slightly* harder than (a) merely because its right-hand side takes slightly longer to evaluate.

Example 2

Numerical results may even be more accurate, in some cases, than the ‘exact’ result:

Find $\int_{999}^{1001} x^2 \log x \, \mathrm{d}x$.

[‘log’ is to base e; same as ‘ln’; never try to do calculus with logs to base 10!]

Ans: $\left[\frac{1}{3}x^3(\log x - \frac{1}{3})\right]_{999}^{1001}$.

Evaluate by calculator! [No traps.]

(a) 13815516.10 ;

(b) compare $2 \times 1000^2 \times \log 1000 = 13815510.56$;

(c) compare Simpson’s Rule,
$(f(999) + 4f(1000) + f(1001))/3 = 13815516.16[3134404\ldots]$;

(d) compare ‘exact’ result [Maple]: 13815516.163134426....

Example 3

Numerical results can certainly quite often be obtained to 'sufficient' accuracy much more easily than 'exact' results:

Find $\int_1^2 \frac{1}{x^4+1}\,dx$.

Ans: $\frac{1}{8}\sqrt{2}\left[\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + 2\tan^{-1}(\sqrt{2}x+1) + 2\tan^{-1}(\sqrt{2}x-1)\right]_1^2$

$= \ldots = 0.2031547019$ [Maple].

How long would that take you? Compare Simpson's Rule:

$$\tfrac{1}{6}[\,f(1)+4f(\tfrac{3}{2})+f(2)] = \tfrac{1}{6}[\tfrac{1}{2}+\tfrac{64}{97}+\tfrac{1}{17}] = 0.20310289\ldots.$$

Example 4

Some apparently simple numerical problems can have unexpected twists.

Find $\int_0^1 x^{15}\,\mathrm{e}^{x-1}\,\mathrm{d}x$.

OK, let $I_n = \int_0^1 x^n\,\mathrm{e}^{x-1}\,\mathrm{d}x$. We can integrate by parts; for $n > 0$, we have

$$I_n = \left[x^n\,\mathrm{e}^{x-1}\right]_0^1 - \int_0^1 n x^{n-1}\,\mathrm{e}^{x-1}\,\mathrm{d}x = 1 - nI_{n-1}.$$

Now we can work up, starting from

$I_0 = \int_0^1 \mathrm{e}^{x-1}\,\mathrm{d}x = \left[\mathrm{e}^{x-1}\right]_0^1 = 1 - 1/\mathrm{e} = 0.632120558\ldots.$

So $I_1 = 1 - 1I_0 = 0.367879442\ldots.$

So $I_2 = 1 - 2I_1 = 0.264241117\ldots.$

So $I_3 = 1 - 3I_2 = 0.20\ldots.$

So ....

So $I_{15} = 1 - 15I_{14} = \ldots.$

Please, before the next lecture use your calculator to find I_{15}. See if you notice anything. [Don't worry if you don't; your calculator may be 'different'.]