Ordinary Differential Equations

Introduction

We look at the ordinary differential equation [ODE]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

given some starting value $y = y_0$ when $x = x_0$.

- Almost everything remains the same if y is a vector, $\boldsymbol{y}$. So we can solve simultaneous ODEs, using exactly the same methods, as long as there is only one independent variable. There can be problems if the one initial condition is replaced by several conditions on components of $\boldsymbol{y}$ at varying values of x ['boundary value problem' as opposed to 'initial value problem'].

- We consider partial differential equations [PDEs], where there are two or more independent variables, later.

- We don't need f to be linear, simple, whatever. But it must be 'well-behaved': continuous, at least. There are also many ways in which f can be 'ill-conditioned', meaning that the solution is rather delicate.

- Second-order equations can be replaced by a vector of first-order equations:

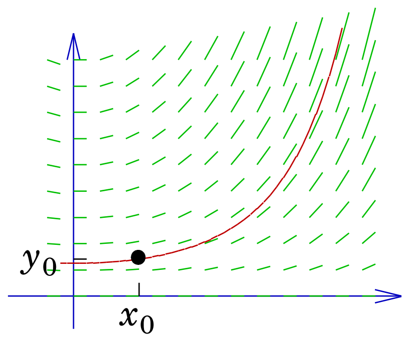
  $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x,y,\frac{\mathrm{d}y}{\mathrm{d}x})$$

  is equivalent to

  $$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_2; \quad \frac{\mathrm{d}y_2}{\mathrm{d}x} = f(x,y_1,y_2).$$

  So we don't need special techniques for these.

Any ODE gives the slope of the solution at each point [for which f is defined:



Our problem is to 'join up' the lines.

[The diagram shows the ODE $dy/dx = xy$, with solution $y = Ae^{\frac{1}{2}x^2}$.]

Euler's method

This was the first and simplest way proposed to solve ODEs. For a given, small, step-length h, we have

$$\frac{dy}{dx} \approx \frac{y(x+h)-y(x)}{h},$$

and so

$$y(x+h) \approx y(x)+h\frac{dy}{dx} = y(x)+hf(x,y).$$

So, we start from $x = x_0$, $y = y_0$, and calculate $k_0 = hf(x_0,y_0)$, $x_1 = x_0+h$, $y_1 \approx y_0+k_0$. Then we move on to $x = x_1$, $y = y_1$, and estimate k_1, x_2, y_2 similarly, and keep going until we have built up as large a table of values of y as we need.

Example

Consider the case $f(x,y) = xy$ sketched above, with initial conditions $y = 1$ when $x = 0$, so that the exact solution is $y = e^{\frac{1}{2}x^2}$. We choose $h = 0.1$. Then we can build up a table:

x	y	k
0.0	1.0000	0.0000
0.1	1.0000	0.0100
0.2	1.0100	0.0202
0.3	1.0302	0.0309
0.4	1.0611	0.0424
0.5	1.1036	0.0552
0.6	1.1587	0.0695
0.7	1.2283	0.0860
0.8	1.3142	0.1051
0.9	1.4194	0.1277
1.0	1.5471	

[numbers given to 4dp, but worked to full calculator accuracy]. The last value is not a million miles from $e^{\frac{1}{2}} \approx 1.6487$, but it is not that marvellous either.

If we try again with $h = 0.01$, we find $y(1) \approx 1.6378$, and with $h = 0.001$, we find $y(1) \approx 1.6476$. [Computer program rather than pencil and paper!] Note that using $h = 0.001$ is a huge amount of work. So Euler's method gives us the choice between doing a reasonable amount of work [comparable with Simpson's Rule, say] and getting a poor result [error roughly 0.1, when we might have hoped for three or four sf] or doing a huge amount of work to get a decent approximation to the right value.

How can we do better? First we need:

Error analysis

Suppose that y has a Taylor series about $x = x_n = x_0 + nh$. That is,

$$y_{n+1} = y_n + h\frac{\mathrm{d}y}{\mathrm{d}x} + \tfrac{1}{2}h^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \ldots.$$

Then Euler's method is giving

$$y_{n+1} \approx y_n + hf(x_n, y_n),$$

and this is wrong for two reasons:

- There is a truncation error, $-\tfrac{1}{2}h^2y''$. Over n strips, this builds up to $-\tfrac{1}{2}(nh)hy''$, where nh tells us how far we have gone, and y'' is some typical second derivative. So the error is proportional to h, which is why the results are not marvellous, but can be made decent by taking *very* thin strips.

- The value of y_n that we are using is not the true value of $y(x_n)$, but is an approximation. This means that we also depend on structural features of the ODE. After a bit, we are solving the ODE as though we had started from slightly the wrong value of y_0. If we are lucky, this will give us slightly the wrong answer; if we are unlucky, then the exact solution will depend very strongly on y_0 [a *stiff* ODE], and our 'wrong' answer will rapidly become *very* wrong.

We can at least quite easily improve Euler's method:

Modified Euler Method

Start as before, $k_1 = hf(x_0, y_0)$. So our Euler guess is $y_1 \approx y_0 + k_1$. Now let $k_2 = hf(x_1, y_1)$. Then the change between k_1 and k_2 is telling us how rapidly f is changing as we change x. That is, it is giving us an approximation to the second derivative that appeared in the truncation error. The *average* of k_1 and k_2 will get rid of that approximate error. In other words, we 'guess' y_1 by the Euler method, use the guess to find k_2, and then produce a 'refined' guess, $y_1 \approx y_0 + \frac{1}{2}(k_1 + k_2)$. If we again take $h = 0.1$, our Modified Euler Method gives:

x	y	k_1	k_2
0.0	1.0000	0.0000	0.0100
0.1	1.0050	0.0101	0.0203
0.2	1.0202	0.0204	0.0312
0.3	1.0460	0.0314	0.0431
0.4	1.0832	0.0433	0.0563
0.5	1.1331	0.0567	0.0714
0.6	1.1971	0.0718	0.0888
0.7	1.2774	0.0894	0.1093
0.8	1.3768	0.1101	0.1338
0.9	1.4988	0.1349	0.1634
1.0	1.6479		

[This is my recommended layout!]

As you can see, this is much better! Again, $y(1)$ should be $e^{\frac{1}{2}} = 1.6487\ldots$, so the error is reduced to about 0.001.

You should *always* use this Modified Euler Method in preference to the unmodified version. If the extra work bothers you, you can double the size of h and still get much better results. In serious work, you should not use the modified scheme either, as there are better ways still ....