Partial Differential Equations

Introduction

Partial differential equations [PDEs] arise when there are two or more independent variables in a differential equation. As with ODEs, there is a very broad grouping into initial value problems ['This is the starting position, how does it evolve?'] and boundary value problems ['This is what is happening at the outside, what is happening in the middle?']. Also as with ODEs, there is a much better prospect of making progress with linear PDEs than with more general non-linear PDEs. And also as with ODEs, there are problems of stability and stiffness. In other words, we get all the problems/notions/ideas of ODEs plus the new ones caused by the extra variable[s]—meaning that boundary points become boundary lines or surfaces, arbitrary constants become arbitrary functions, and so on, but also that there are extra stability problems caused by potential interference between the variables.

Most PDEs arising in practice are of the second order. The simplest/commonest examples are Laplace's equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

[or Poisson's equation, similar but with non-zero RHS], the diffusion equation, or heat equation:

$$\frac{\partial\phi}{\partial t} = k\frac{\partial^2\phi}{\partial x^2},$$

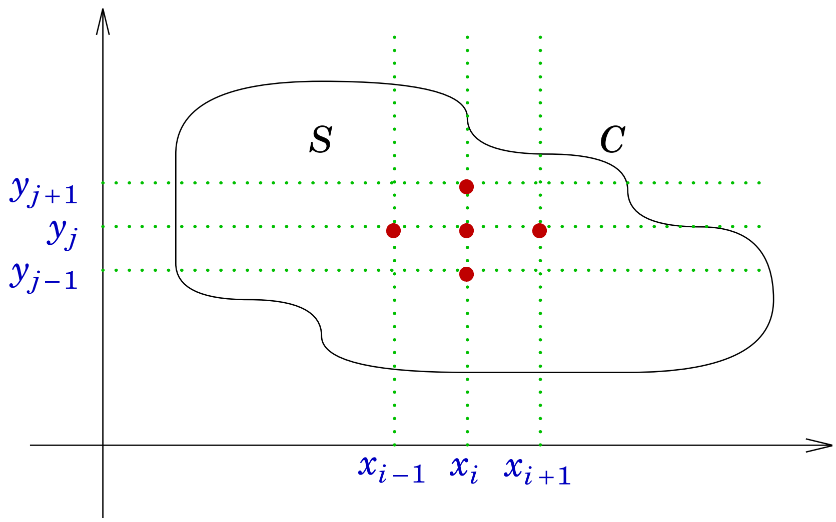
and the wave equation:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}.$$

We take these in turn, as they present different problems for numerical solution.

Elliptic† PDEs—Laplace's Equation

Laplace's equation, $\nabla^2\phi = 0$, is usually presented as a BVP.



The assumption is that the equation holds over some region S, with $\phi(x,y)$ prescribed on the boundary C. [The equation could, for example, describe—among many other applications—the displacement of a membrane, such as a drum-skin, which is in equilibrium and clamped in position round the edge.]

Following our procedure with BVPs in ODEs, we can replace the PDE by finite-difference equations. We choose some strip width h for the x-values, and k for the y-values. So we divide S into little rectangles, bordered by lines $x = x_i$ running vertically and y_j running horizontally. We assume that where these lines meet, we can find an approximate value $\phi(x_i, y_j) \approx u_{i,j}$, from which we can build up a matrix of values, and we need one equation for each combination of i and j.

† Elliptic, parabolic, hyperbolic PDEs—classification discussed later.

This comes from essentially the same equation as we used for ODEs:

$$\frac{\partial^2\phi}{\partial x^2} \approx (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/h^2,$$

and correspondingly for y,

$$\frac{\partial^2\phi}{\partial y^2} \approx (u_{i,j-1} - 2u_{i,j} + u_{i,j+1})/k^2.$$

So Laplace's equation becomes

$$(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/h^2 + (u_{i,j-1} - 2u_{i,j} + u_{i,j+1})/k^2 \approx 0,$$

$$u_{i,j} \approx \frac{k^2}{2h^2+2k^2}(u_{i-1,j} + u_{i+1,j}) + \frac{h^2}{2h^2+2k^2}(u_{i,j-1} + u_{i,j+1}),$$

relating the u-values marked by dots in the diagram. Note that $u_{i,j}$ is a weighted average of the four surrounding values; in the common case where $h = k$, it is the simple average.

At or very near the boundary, C, this simple relationship breaks down if C is at all irregular, and some tweaking is needed. If C is rectangular, then, as with ODEs, the effect is that some of the u-values are already known.

- Note that we now get large numbers of equations even for quite modest numbers of strips—100 or so for about 10 strips in each direction, 1000 if we use the same approach in three dimensions, 10^6 if we have a six-dimensional problem [perhaps in phase space]—so this is definitely a problem for the computer rather than pencil-and-paper except for toy examples.

- No particularly new principles if we tackle Poisson's equation, $\nabla^2\phi = f(x,y,\phi)$; there will be a non-zero, and perhaps non-linear, term on the RHS of the equations, so iteration may be needed. Also no new principles if we change, say, to polar co-ordinates.

Toy Example

Suppose S is the square $0 \leqslant x,y \leqslant 1$, and that on C, $\phi = x^2$ on $y = 0$, $\phi = y^2$ on $x = 0$, $\phi = 1$ on $x,y = 1$. Choose $h = k = 0.25$, so there are nine interior points. By symmetry, $u_{i,j} = u_{j,i}$, so we only need six equations, and also $u_{i,j}$ is known if either of i or j is 0 or 4. So we get:

$$u_{1,1} \approx \tfrac{1}{4}(u_{0,1}+u_{2,1}+u_{1,0}+u_{1,2}) = \tfrac{1}{2}(\tfrac{1}{16}+u_{1,2});$$

$$u_{1,2} = u_{2,1} \approx \tfrac{1}{4}(u_{0,2}+u_{2,2}+u_{1,1}+u_{1,3}) = \tfrac{1}{4}(\tfrac{1}{4}+u_{2,2}+u_{1,1}+u_{1,3});$$

$$u_{1,3} = u_{3,1} \approx \tfrac{1}{4}(u_{0,3}+u_{2,3}+u_{1,2}+u_{1,4}) = \tfrac{1}{4}(\tfrac{25}{16}+u_{2,3}+u_{1,2});$$

$$u_{2,2} \approx \tfrac{1}{4}(u_{1,2}+u_{3,2}+u_{2,1}+u_{2,3}) = \tfrac{1}{2}(u_{1,2}+u_{2,3});$$

$$u_{2,3} = u_{3,2} \approx \tfrac{1}{4}(u_{1,3}+u_{3,3}+u_{2,2}+u_{2,4}) = \tfrac{1}{4}(1+u_{1,3}+u_{3,3}+u_{2,2});$$

$$u_{3,3} \approx \tfrac{1}{4}(u_{2,3}+u_{4,3}+u_{3,2}+u_{3,4}) = \tfrac{1}{2}(1+u_{2,3}).$$

We can use Maple to solve these:

$$u_{1,1} \approx 0.265,\ u_{1,2} = u_{2,1} \approx 0.466,\ u_{1,3} = u_{3,1} \approx 0.711,$$
$$u_{2,2} \approx 0.641,\ u_{2,3} = u_{3,2} \approx 0.815,\ u_{3,3} \approx 0.907.$$

The values for the exact solution of the PDE would have been [to 3sf]

$$u_{1,1} \approx 0.271,\ u_{1,2} = u_{2,1} \approx 0.473,\ u_{1,3} = u_{3,1} \approx 0.716,$$
$$u_{2,2} \approx 0.647,\ u_{2,3} = u_{3,2} \approx 0.819,\ u_{3,3} \approx 0.910.$$

Beyond the scope of this module, the truncation error is of order h^2+k^2 and is also proportional to a rather messy function of the fourth partial derivatives of ϕ [so in fact we get exact answers if the true solution is ‘only’ cubic in x and y].