Non-linear equations

Iterative methods

These generalise the basic idea we saw with Newton-Raphson. We have some approximation to the solution, and we apply some process [‘algorithm’] to the approximation in order to generate [with luck] a better approximation. We go round and round [which is what ‘iteration’ implies] until the approximation is ‘good enough’.

Mathematically, this is usually written as some sort of sequence:

$$x_{n+1} = g(x_n)$$

for some function g related to f [for Newton-Raphson, it was $g(x) = x - f(x)/f'(x)$] and for some initial approximation x_0. For computing purposes, you won’t need the sequence, we can just assign some new value to x:

$$x \rightarrow g(x)$$

[but using whatever notation your preferred language requires].

How to generate/invent the function g? Newton-Raphson is one way. But basically, we can tweak f in any way we like from the form $f(x) = 0$ into the form $g(x) = x$ to give a potential iteration. It could be as simple as adding x to both sides! But we can usually be more inventive.

Examples

Wallis:

[The first four are obtained just by isolating one term of the equation and 'solving' it for x; the fifth is Newton-Raphson.]

$$x \;\to\; (2x+5)^{1/3};$$

$$x \;\to\; \tfrac{1}{2}(x^3-5);$$

$$x \;\to\; (2x+5)/x^2;$$

$$x \;\to\; \sqrt{2+5/x};$$

$$x \;\to\; x - \frac{x^3-2x-5}{3x^2-2} \;=\; \frac{2x^3+5}{3x^2-2}.$$

Tan:

[Again, various obvious tweakings followed by Newton-Raphson.]

$$x \;\to\; \tan x;$$

$$x \;\to\; \tan^{-1} x;$$

$$x \;\to\; \tfrac{1}{2}(x + \tan x);$$

$$x \;\to\; \sqrt{x \tan x};$$

$$x \;\to\; x - \frac{\tan x - x}{\sec^2 x - 1} \;=\; \frac{x\sec^2 x - \tan x}{\tan^2 x}.$$

Try these! Start from reasonable values, such as 2 [Wallis] or 4.5 [Tan].

We can investigate convergence by assuming that g has a Taylor series:

$$g(x) = g(p) + (x-p)g'(p) + \tfrac{1}{2}(x-p)^2 g''(p) + \ldots .$$

If p is the root, such that $g(p) = p$, this gives

$$x_{n+1} = g(x_n) = g(p) + (x_n - p)g'(p) + \tfrac{1}{2}(x_n - p)^2 g''(p) + \ldots ,$$

or

$$e_{n+1} = g'(p)e_n + \tfrac{1}{2}g''(p)e_n^2 + \ldots ,$$

where $e_n = x_n - p$ is the error on the nth iteration.

So, it is *good* to have $g'(p)$, $g''(p)$, ... small, and *bad* to have them large.

Special cases:

(a) $|g'(p)| > 1$: there is no convergence; small errors grow. $|g'(p)| = 1$ is also Bad News.

(b) $0 < |g'(p)| < 1$: *linear* convergence [also called geometric or first-order], as long as x_0 is close enough to p [so that the truncation error $\tfrac{1}{2}g''(p)e_0^2 + \ldots$ is not too large].

(c) $|g'(p)| = 0$: *quadratic* [or second-order] convergence as long as x_0 is close enough to p. The error is roughly squared [Good News!] if it is small. Even better if $g''(p) = 0$.