Difference Tables

In pre-computer days, these were the principal tool of numerical work; indeed, their automation was the 'inspiration' for Babbage's 'Difference Engine' and 'Analytical Engine', now widely regarded as the [mid-19th C] forerunner of the whole idea of automatic computing. They are no longer used 'in anger' in pure numerical work; but they are still very useful in preliminary analysis [especially by pencil and paper] of experimental results.

Start with an example. Here is the Wallis polynomial, $W(x) = x^3 - 2x - 5$, for $x = -2, -1, \ldots, 8$.

x	$W(x)$				
–2	–9				
		5			
–1	–4		–6		
		–1		6	
0	–5		0		0
		–1		6	
1	–6		6		0
		5		6	
2	–1		12		0
		17		6	
3	16		18		0
		35		6	
4	51		24		0
		59		6	
5	110		30		0
		89		6	
6	199		36		0
		125		6	
7	324		42		
		167			
8	491				

[After the $W(x)$ column, each number is the difference between the number left and below and the number left and above.]

Notes:

(a) In this particular case, the third column of differences, the ‘third differences’, are constant; so fourth differences are zero, as are fifth, sixth, *etc*. This is much the same result as that, for this particular function, the third *derivative* is constant, and fourth and higher derivatives zero.

(b) If we know any of the diagonal lines in the table, or enough of the table to construct one, then we can fill in the rest of the table. This is, in fact, the easiest way, in general, to build up a table of polynomial values For example, $W(9) = 491 + 167 + 42 + 6 = 706$.

(c) In this table, we had a cubic polynomial, and we tabulated it at intervals [for x] of 1 between rows. *Exercise:* Verify by example that if we difference *any* polynomial, at *any* interval [as long as it is the same throughout the table†] each successive column gives the value of a polynomial of one lower degree, until the nth differences are constant and further differences zero, where n is the degree of the polynomial.

(d) But the converse of this result is not true. Things that difference [ultimately] to zero are not necessarily polynomial. Specifically, we can take any function $G(x)$ whatsoever; then $G(x) \sin \pi x$ is zero whenever x is an integer, and so $W(x) + G(x) \sin \pi x$ would have had exactly the same table, but is in general a quite difference function. To make progress, we need to assume that the table ‘fairly represents’ the values of [in this case] $W(x)$.

(e) But if we ‘play fair’, then we can get more from a difference table than you might expect:

† It is possible, but much, much harder, to do NA on tables that are not at constant interval. You really, really do not want to know. [If you do, look up ‘divided differences’.]

Example

Here is a table of values of $\sin x$, at a ‘tabular interval’ of 0.2 in the values of x.

x	$\sin x$						
0.0	0.00000						
		19867					
0.2	0.19867		–792				
		19075		–761			
0.4	0.38942		–1553		64		
		17522		–697		22	
0.6	0.56464		–2250		86		*10*
		15272		–611		32	
0.8	0.71736		–2861		118		*–19*
		12411		–493		13	
1.0	0.84147		–3354		131		*5*
		9057		–362		18	
1.2	0.93204		–3716		149		*–9*
		5341		–213		9	
1.4	0.98545		–3929		158		*–13*
		1412		–55		–4	
1.6	0.99957		–3984		154		
		–2572		99			
1.8	0.97385		–3885				
		–6457					
2.0	0.90928						

(f) By convention, we write everything after the actual values of $\sin x$ as an integer; this saves lots of writing of 0.0000, and makes the arithmetic easier, *as long as* you write out the table neatly! You just have to remember to put the decimal point back in when you produce actual function values.

(g) Note that the differences [sort-of] get smaller as we go to second, third, fourth differences—especially remembering that they are really 0.000-something—but they are never zero. This is partly because they really aren't zero ['truncation error'], and if we did everything exactly they still wouldn't be zero; but much more because of rounding errors. See the next table:

$\frac{1}{2}$						
	–1					
$-\frac{1}{2}$		2				
	1		–4			
$\frac{1}{2}$		–2		8		
	–1		4		–16	
$-\frac{1}{2}$		2		–8		32
	1		–4		16	
$\frac{1}{2}$		–2		8		–32
	–1		4		–16	
$-\frac{1}{2}$		2		–8		
	1		–4			
$\frac{1}{2}$		–2				
	–1					
$-\frac{1}{2}$						

This shows how an alternating error of half a unit in the last decimal—the worst possible rounding in a correct table—builds up by powers of two.

Errors as large as $\pm 2^{n-1}$ in the nth differences *could* be due entirely to chance rounding errors. You should completely ignore values smaller than this—they are just 'noise', and using them will make your results worse rather than better.

In the case of the sine table, we see that fifth differences are not [all] as small as ± 16, but sixth differences [in italics] are all smaller than ± 32. *Over this range of values of x and for this tabular interval [0.2],* $\sin x$ *to 5dp behaves very like a polynomial of degree 5; this observation 'guides' our choice of further numerical techniques on these results.*

(h) If some function [eg the result of an experiment] does *not* behave like a polynomial, what then? Three possible causes: *(i)* The tabular interval is too large, so the table does not ‘capture’ the function adequately. [Bit late if this is an expensive experiment!] You really, really do not have the information to do good numerical work on these figures. *(ii)* The function really is not a polynomial. Almost all numerical methods are designed to work on polynomials; you really need to re-think the function before continuing. [Eg, take logs to get rid of ‘exponential’ behaviour.] *(iii)* One or more of the values were mis-calculated or mis-transcribed, eg when copying experimental results using pencil-and-paper. *Moral:* get these processes automated! Luckily, this last case can easily be detected, and often corrected:

	0		0		1
0		0		1	
	0		1		–5
0		1		–4	
	1		–3		10
1		–2		6	
	–1		3		–10
0		1		–4	
	0		–1		5
0		0		1	
	0		0		–1

The table shows a ‘blunder’ of size 1 in a function that is otherwise zero. Note that the table shows a characteristic pattern of binomial coefficients of alternating sign. If you make a mistake, then the table you get is the table you should have plus some multiple of the table above.

Exercise: Make the ‘blunder’ $\sin 1 = 0.84174$ in the sine table, and re-calculate the table [not all of it!]. Note how easy it is to spot the mistake [transposing digits], to see how big the mistake is [by comparison with the above], and therefore to correct it. *Moral*: It is always worth differencing tables of experimental results.