Mock Exam—Solutions

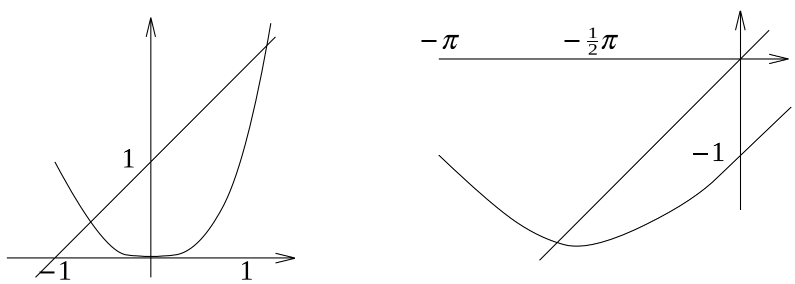
1 (*a*) The roots of the quadratic are

$$x_{1,2} = (A \pm \sqrt{A^2 - 4})/2;$$

for A large, so that A^2 is much greater than 4, the larger root is approximately A, and, since the product of the roots is 1, the smaller root is approximately $1/A$. Both roots will be calculated to the same number of decimal places, but the smaller root will have many fewer significant figures. For example, if $A \approx 2000$, then $1/A \approx 0.0005$, and a calculator working to 12sf will give both roots to 8dp, but whereas the larger will [therefore] have 12sf, the smaller will have only 5sf, and 7sf have been lost.

In the case $A = 483$, the larger root is $(483 + \sqrt{233285})/2 = 482.997929598$ [to 12sf] and therefore the smaller root is $1/482.99\ldots = 0.00207040224962$ [12sf].

(*b*)



(*i*) (ii)

(*i*) The equation $x^4 = x + 1$ clearly [from the sketch] has two real roots, one just greater than -1, the other just greater than $+1$. Newton–Raphson, using $x \leftarrow x - (x^4 - x - 1)/(4x^3 - 1) = (3x^4 + 1)/(4x^3 - 1)$, should be suitable.

Starting from $x = -1$, we find $-1 \to -\frac{4}{5} \to -0.7312335958 \to -0.7245484800 \to -0.7244919630$, and starting from $x = +1$, we find $1 \to \frac{4}{3} \to 1.2358078603 \to 1.2210589943 \to 1.2207442258$. [No need to go round again, except as a check; recall that the error is roughly squared on each iteration, so we only need to keep going until two values agree to 3dp.] The real roots are -0.72449 and 1.22074 to 5dp. [Maple gives -0.7244919590 and 1.2207440846 to 10dp. Note that if you were to need the complex roots, then because the total sum of the roots is zero, and product 1, the complex roots are those of $x^2 + (1.22\ldots - 0.72\ldots)x - 1/(1.22\ldots \times 0.72\ldots) = 0$.]

(*ii*) The equation $x = \sin x - 1$ clearly [from the sketch] has one real root, at around $x = -\frac{1}{2}\pi$. The iteration $x \leftarrow \sin x - 1$ should work well, especially as the RHS is 'flat' at $x = -\frac{1}{2}\pi$. We find $-\frac{1}{2}\pi \to -2 \to -1.9092974268 \to -1.943\ldots \to \ldots \to -1.9345624101$ [12 iterations]. So the root is -1.93456 to 5dp [Maple gives -1.9345632108 to 10dp]. Either Aitken's device or noting that the change is multiplied by around $-\frac{1}{3}$ each time round can be used to accelerate convergence; but the iteration [4 key presses each time] is so simple that there is quite little point. [If you got the answer -1.01776, it is because your calculator was set to degrees.]

2 (*a*) [Details omitted.] We find $I_2 = \frac{1}{6}(1/1+4/\frac{17}{16}+1/2) \approx 0.8774509804$, $I_4 \approx 0.8671137005$ and $I_8 \approx 0.8669810489$. The estimated errors are $(I_2-I_4)/15 \approx 0.0007$ and $(I_4-I_8)/15 \approx 0.00001$, so this is converging very satisfactorily and our estimate is $I = 0.86697$ to 5dp. [Maple gives 0.8669729870. Simpson's Rule gives $I_{16} \approx 0.8669734957$, and then $(I_8-I_{16})/15 \approx 0.0000005035$, giving an estimate of 0.8669729922; you do not need this!]

(*b*) We should not use Simpson's Rule directly because $\sqrt{\sin x} \approx \sqrt{x}$ when x is small, so is badly-behaved at $x = 0$. The substitution $x = u^2$ gives

$$\int_0^1 \sqrt{\sin x}\,\mathrm{d}x = \int_0^1 \sqrt{\sin u^2}\,2u\,\mathrm{d}u,$$

which is well-behaved throughout the interval. [Not asked for, but the actual integral is 0.6429776347 to 10dp; SR with 8 strips gives 0.6429518 (or, with Richardson extrapolation, 0.6429768) to 7dp for the u-integral, but 0.6394 to 4dp for the x-integral [which needs 1024 strips to get the result correct to 5dp.]

3 (*a*) We can build up the table:

1	–1	2	–1	–8	*a*
2	–2	3	–3	20	*b*
1	1	1	0	–2	*c*
1	–1	4	3	4	*d*
0	0	–1	–1	36	$e = b-2a$
0	2	–1	1	6	$f = c-a$
0	0	2	4	12	$g = d-a$
0	0	0	2	84	$h = g+2e$

From (h), we have $w = 42$, then from (e), $z = -78$, then from (f), $2y+120 = 6$, so $y = -57$, and finally from (a), $x = y-2z+w-8 = 133$.

None of the row operations above have changed the determinant of the corresponding matrix. If we think of equations (a), (e), (f) and (h), then swap (e) and (f) [which changes the sign of the determinant], we have a matrix with zeros below the main diagonal, and 1, 2, −1, 2 on that diagonal. So the determinant is $-1\times2\times-1\times2 = 4$.

(*b*) Bookwork—see lecture 13, *pp*. 4 and 5.

4 (*a*) Bookwork—see lecture 14, *p*. 5.

(*b*) Bookwork—see lecture 16, *p*. 2. [You are not expected to reproduce all the numbers, of course!]

(*c*) The equation can be re-cast as a pair of first-order ODEs:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -p(x)z - q(x), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = z.$$

[Oops! As set, the ODE no longer depends on y, so we can now, in principle, solve the first equation for z, and then integrate to find y. The intention was that the first equation should still depend on y explicitly, so that the equations have to be solved together.] We can now use Modified Euler, except that y is replaced by the vector $\boldsymbol{y}$ whose components are y and z, and f (the derivative) by the vector $\boldsymbol{f}$ whose components are z and (in this case) $-p(x)z - q(x)$.

As the equation is linear (in y and z), we can solve the BVP by finding two independent solutions, for example y_1 given by initial conditions $y = 1$, $z = 0$ when $x = 0$, and y_2 given by initial conditions $y = 0$, $z = 1$ when $x = 0$. Then the general solution is $y = \alpha y_1 + \beta y_2$; The boundary conditions then give two simulaneous equations for α and β.

5 (*a*) Bookwork—see lecture 19, especially *p*. 2.

(*b*) In the case $h = k = \frac{1}{3}$, we have [in general]

$$u_{i,j+1} = u_{i,j} + 3(u_{i-1,j} - 2u_{i,j} + u_{i+1,j)} = 3u_{i-1,j} - 5u_{i,j} + 3u_{i+1,j},$$

where $u_{i,j} \approx \phi(i/3, j/3)$. We are concerned only with $0 \leqslant i,j \leqslant 3$, and the boundary conditions give $u_{i,j}$ when $i = 0$ or $i = 3$, and the initial conditions give $u_{i,j}$ when $j = 0$. So the remaining equations are

$$\begin{aligned} u_{1,1} &= 3u_{0,0} - 5u_{1,0} + 3u_{2,0}, \quad u_{2,1} = 3u_{1,0} - 5u_{2,0} + 3u_{3,0}; \\ u_{1,2} &= 3u_{0,1} - 5u_{1,1} + 3u_{2,1}, \quad u_{2,2} = 3u_{1,1} - 5u_{2,1} + 3u_{3,1}; \\ u_{1,3} &= 3u_{0,2} - 5u_{1,2} + 3u_{2,2}, \quad u_{2,3} = 3u_{1,2} - 5u_{2,2} + 3u_{3,2}. \end{aligned}$$

(*c*) In the given case, $u_{0,j} = u_{3,j} = 0$ and $u_{i,0} = i \times (3 - i)/9$, so $u_{1,0} = u_{2,0} = \frac{2}{9}$. Substituting in the above equations, we find

$$u_{1,1} = u_{2,1} = -\tfrac{4}{9}; \quad u_{1,2} = u_{2,2} = \tfrac{8}{9}; \quad u_{1,3} = u_{2,3} = -\tfrac{16}{9}.$$

[As noted in the lecture, this solution technique is unstable unless we take k much smaller.]

End